Section 4.4 (2^{nd} Ed.) or 6.3 (1^{st} Ed.): Change of basis

Definition: Let $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_k\}$ be a basis of a subspace S of $\mathbb{R}^n$ (which can be all of R^n), and let $\vec{w}$ be a vector in S. The *coordinates* of $\vec{w}$ in the basis $\mathcal{B}$ are the k numbers $a_1, a_2, \ldots, a_k$ such that

$$\vec{w} = a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_k\vec{u}_k.$$

NOTATION: The coordinates of a vector $\vec{w}$ in a basis $\mathcal{B}$ will be denoted $[\vec{w}]_{\mathcal{B}}$.

Example: Consider the basis of $\mathbb{R}^2$ given by $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$.

Then the vector $\vec{w}$ with coordinates $[\vec{w}]_{\mathcal{B}} = (2, 3)$ is

$$2\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-3 \\ 4+3 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}$$

Another example: what are the coordinates of the vector $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ in the basis $\mathcal{B}$? We would need to find x_1, x_2 such that

$$x_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

Do it as usual:

$$\left(\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & 1 & 5 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 3 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 1 & -1/3 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 8/3 \\ 0 & 1 & -1/3 \end{array}\right)$$

The answer is, therefore, $\left[\begin{pmatrix} 3 \\ 5 \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} 8/3 \\ -1/3 \end{pmatrix}$. It is easy to check that this is correct:

$$\frac{8}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

Exercises: Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$ be a basis of some subspace S of R^3.

- Find the vector $\vec{w} \in S \subset \mathbb{R}^3$ whose coordinates in the basis $\mathcal{B}$ are

$$\begin{pmatrix} 5 \\ -3 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- The following vectors are in S. Find their coordinates in the basis $\mathcal{B}$:

$$\vec{u} = \begin{pmatrix} 3 \\ 9 \\ -1 \end{pmatrix}. \qquad \vec{v} = \begin{pmatrix} 7 \\ -1 \\ 5 \end{pmatrix}. \qquad \vec{w} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Change of coordinates between two bases of $\mathbb{R}^n$

Recall: The standard basis of $\mathbb{R}^n$ is the basis st $= \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ given by

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

The standard basis will be denoted by "st", and the coordinates of a vector $\vec{w}$ in the standard basis will be denoted by $[\vec{w}]_{\text{st}}$

Observation: Any vector $\vec{w} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ in $\mathbb{R}^n$ can be written, in the standard basis, as $a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n$, so the coordinates of $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ in the standard basis are just $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$. In other words, $[\vec{w}]_{\text{st}} = \vec{w}$.

<u>Changing from a basis of $\mathbb{R}^n$ to the standard basis</u>

Suppose that the coordinates of a vector $\vec{w}$ in the basis $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_n\}$ of $\mathbb{R}^n$, are $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, that is, $[\vec{w}]_{\mathcal{B}} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$.

What are the coordinates of $\vec{w}$ in the standard basis?
This is not hard: we have $\vec{w} = a_1\vec{u}_1 + \cdots + a_n\vec{u}_n$, and therefore by the previous observation, $[\vec{w}]_{\text{st}} = a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_n\vec{u}_n$. This can be written as

$$[\vec{w}]_{\text{st}} = [\vec{u}_1, \vec{u}_2, \cdots \vec{u}_n] \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = U[\vec{w}]_{\mathcal{B}}.$$

We can therefore write

$$\boxed{[\vec{w}]_{\text{st}} = U\,[\vec{w}]_{\mathcal{B}}, \text{ where } U = [\vec{u}_1, \vec{u}_2, \cdots \vec{u}_n].}$$

<u>Changing from the standard basis to another basis of $\mathbb{R}^n$</u>

In view of the last equation, this is easy: multiply the last equation on both sides by U^{-1} on the left to get

$$\boxed{[\vec{w}]_{\mathcal{B}} = U^{-1}[\vec{w}]_{\text{st}}, \text{ where } U = [\vec{u}_1, \vec{u}_2, \cdots \vec{u}_n].}$$

<u>Changing coordinates from a basis $\mathcal{B}$ of $\mathbb{R}^n$ to another basis $\mathcal{C}$ of $\mathbb{R}^n$</u>

Suppose that $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \cdots \vec{u}_n\}$ and $\mathcal{C} = \{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n\}$ are bases. If $\vec{w}$ is in $\mathbb{R}^n$, then we have

$$[\vec{w}]_{\mathcal{C}} = V^{-1}[\vec{w}]_{\text{st}} \text{ and } [\vec{w}]_{\text{st}} = U\,[\vec{w}]_{\mathcal{B}}.$$

Substituting the second equation into the first, we get

$$\boxed{[\vec{w}]_{\mathcal{C}} = V^{-1}U\,[\vec{w}]_{\mathcal{B}}, \quad \text{where } U = [\vec{u}_1, \vec{u}_2, \cdots \vec{u}_n] \text{ and } V = [\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n].}$$

It is hard to keep track of this, so we will use the following

Notation: the matrix of change of coordinates from a basis $\mathcal{B}$ to a basis $\mathcal{C}$ will be denoted by $[I]_{\mathcal{C},\mathcal{B}}$. It will be called **the matrix of change of basis from a basis $\mathcal{B}$ to a basis $\mathcal{C}$ with the standard basis**. We can then write

$$\boxed{\begin{array}{c}[\vec{w}]_{\mathcal{C}} = [I]_{\mathcal{C},\mathcal{B}}\,[\vec{w}]_{\mathcal{B}}, \quad \text{where} \quad [I]_{\mathcal{C},\mathcal{B}} = V^{-1}U, \\ \text{with} \quad U = [\vec{u}_1, \vec{u}_2, \cdots \vec{u}_n] \quad \text{and} \quad V = [\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n].\end{array}}$$

Note that the matrices of change of coordinates have the following intuitive rules. If you remember the first 3, things get easier:

1. $[I]_{\text{st},\mathcal{B}} = U$.
2. $[I]_{\mathcal{B},\mathcal{C}} = ([I]_{\mathcal{C},\mathcal{B}})^{-1}$
3. If we have 3 bases, $[I]_{\mathcal{A},\mathcal{C}} = [I]_{\mathcal{A},\mathcal{B}}\,[I]_{\mathcal{B},\mathcal{C}}$.

For example, to remember the formula $[I]_{\mathcal{C},\mathcal{B}} = V^{-1}U$, you can write

$$[I]_{\mathcal{C},\mathcal{B}} = [I]_{\mathcal{C},\text{st}}\,[I]_{\text{st},\mathcal{B}} = ([I]_{\text{st},\mathcal{C}})^{-1}\,[I]_{\text{st},\mathcal{B}} = V^{-1}U.$$

Examples: Let $\mathcal{B} = \{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}4\\3\end{pmatrix}\}$ and $\mathcal{C} = \{\begin{pmatrix}-1\\4\end{pmatrix}, \begin{pmatrix}2\\-7\end{pmatrix}\}$. If the coordinates of $\vec{w}$ in the basis $\mathcal{B}$ are $\begin{pmatrix}-4\\5\end{pmatrix}$, find the coordinates of $\vec{w}$ in the basis $\mathcal{C}$ and in the standard basis.
Solution: we know $[\vec{w}]_{\mathcal{B}} = \begin{pmatrix}-4\\5\end{pmatrix}$. We want to find $[\vec{w}]_{\mathcal{C}}$ and $[\vec{w}]_{\text{st}}$.
$[\vec{w}]_{\text{st}} = [I]_{\text{st},\mathcal{B}}\,[\vec{w}]_{\mathcal{B}}$. Also, $[I]_{\text{st},\mathcal{B}} = \begin{pmatrix}1 & 4\\2 & 3\end{pmatrix}$. Therefore $[\vec{w}]_{\text{st}} = \begin{pmatrix}1 & 4\\2 & 3\end{pmatrix}\begin{pmatrix}-4\\5\end{pmatrix} = \begin{pmatrix}16\\7\end{pmatrix}$.
$[\vec{w}]_{\mathcal{C}} = [I]_{\mathcal{C},\mathcal{B}}\,[\vec{w}]_{\mathcal{B}} = [I]_{\mathcal{C},\text{st}}\,[I]_{rmst,\mathcal{B}}\,[\vec{w}]_{\mathcal{B}} = ([I]_{\mathcal{C},\text{st}})^{-1}\,[I]_{\mathcal{B},rmst}\,[\vec{w}]_{\mathcal{B}}$.
Now, $[I]_{\mathcal{C},\text{st}} = \begin{pmatrix}-1 & 2\\4 & -7\end{pmatrix}$, so $([I]_{\mathcal{C},\text{st}})^{-1} = \begin{pmatrix}7 & 2\\4 & 1\end{pmatrix}$.
Therefore, $[I]_{\mathcal{C},\mathcal{B}} = \begin{pmatrix}7 & 2\\4 & 1\end{pmatrix}\begin{pmatrix}1 & 4\\2 & 3\end{pmatrix} = \begin{pmatrix}11 & 34\\6 & 19\end{pmatrix}$, and $[\vec{w}]_{\mathcal{C}} = \begin{pmatrix}11 & 34\\6 & 19\end{pmatrix}\begin{pmatrix}-4\\5\end{pmatrix} = \begin{pmatrix}126\\71\end{pmatrix}$

Exercises
For the given bases, find the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$.

- $\mathcal{B} = \left\{\begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}-2\\-5\end{pmatrix}\right\}$, $\mathcal{C} = \left\{\begin{pmatrix}4\\5\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right\}$
- $\mathcal{B} = \left\{\begin{pmatrix}-4\\3\end{pmatrix}, \begin{pmatrix}1\\-2\end{pmatrix}\right\}$, $\mathcal{C} = \left\{\begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\}$
- $\mathcal{B} = \left\{\begin{pmatrix}1\\3\\2\end{pmatrix}, \begin{pmatrix}-2\\-5\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\3\end{pmatrix}\right\}$, $\mathcal{C} = \left\{\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix} \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$
- $\mathcal{B} = \left\{\begin{pmatrix}5\\2\end{pmatrix}, \begin{pmatrix}3\\-2\end{pmatrix}\right\}$, $\mathcal{C} = \left\{\begin{pmatrix}0\\-4\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right\}$