MTH 42 Final Exam Review

Topical Outline

Chapter 1

- Elementary row operations and row reduction.
- Echelon form and reduced row echelon form.
- Consistent and inconsistent systems of equations.
- Parametric descriptions of solutions.

Chapter 2

- Vector equations.
- Linear combinations and span.
- Homogeneous systems, nontrivial solutions, geometric interpretation.
- Linear dependence and independence (definitions, be able to connect this back to solutions to matrix equations, pivots, span, etc.).

Chapter 3

- Linear transformations:
  - Definition of linear transformation.
  - Test if a transformation is linear.
  - Turn a transformation into a matrix.
  - One-to-one and onto transformations.
  - Theorem: T is one-to-one iff $T(x) = 0$ has only the trivial solution.
  - Theorem: T is onto iff the columns of associated matrix A span the codomain.
  - Theorem: T is one-to-one iff the columns of A are linearly independent.
- Matrix operations (add, scalar multiply, multiply, transpose); properties of matrix multiplication.
- Transposes and their properties.
- Matrix inversion (2 by 2 or 3 by 3 by hand).
- Characterizations of invertible matrices.

Chapter 4 sections

- Definition of a subspace of $\mathbb{R}$.
- Examples of subspaces.
- Verifying that a set is (or is not) a subspace.
- Find the null space, column space, row space of a matrix.
- Definition of a basis of a vector subspace, definition of the dimension of a vector subspace.
- Definition of the rank of a matrix.
- How to find the coordinate of a vector with respect to a given basis.
- Change of basis.
- Rank-nullity theorem.

Chapter 5

- Definition of determinant; properties of determinants.

- Compute determinants.
- Calculating 2 by 2, 3 by 3, and 4 by 4 determinants by hand.
- Properties of determinants.
- Applications of determinants: Cramer's rule; inversion of matrices; area and volume.

Chapter 6

- Definitions of eigenvalues/eigenvectors/eigenspaces.
- The geometry of eigenvectors and eigenvalues.
- Calculating eigenvalues ($det(A - \lambda I) = 0$).
- Independence of eigenvectors associated with distinct eigenvalues.
- The characteristic equation.
- Similarity transformations with eigenvectors / values (that is, complex eigenvectors).
- Diagonalization of matrices.
- Diagonalization theorems with distinct and non-distinct eigenvalues (algebraic and geometric multiplicity).

Chapter 7

- Definition of a (abstract) vector space.
- Examples of vector spaces.
- Subspaces of vector spaces.
- Verifying that a set is (or is not) a subspace.
- Definition of span and linear independence in a vector space.
- Definition of a basis of a vector space, definition of the dimension of a vector space.
- How to find the coordinate of a vector with respect to a given basis.
- Change of basis.

Chapter 8

- Definition of an inner (or dot) product.
- Length (norm) and distance in $\mathbb{R}^n$.
- Orthogonality of vectors (inner product equals zero).
- Orthogonal projections/ complements.
- Definition of orthogonal sets / orthogonal bases/ orthonormal bases.
- Unit vectors
- Orthogonal projections and decomposing a vector into orthogonal parts.
- Orthonormal bases
- Orthogonal decomposition theorem
- The Gram-Schmidt Process

Review Problems

1. Review Midterm Review Problems.
2. True or False? You do not need to justify your answers.
   (a) If $A \in \mathbb{M}_{m\times n}(\mathbb{R})$, then the set of solutions of a linear system $A\mathbf{x} = \mathbf{b}$ must be a linear subspace of $\mathbb{R}^n$.

(b) If A and B are invertible $n \times n$ matrices, then so is $A + B$.

(c) If A and B are $n \times n$ matrices, then $A^2 = B^2$ implies $A = B$ or $A = -B$.

(d) If A is a square matrix and $\det A = 7$ then $\det(2A) = 14$.

(e) An elementary row operation on a square matrix A does not change the determinant.

(f) There exists a 2×3 matrix with nullity 3.

(g) There does not exist a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that kernel of $T$and image of T are both lines in $\mathbb{R}^3$.

(h) If A is similar to B then $\det A = \det B$.

(i) If $\det A = \det B$ then A is similar to B.

(j) If a 5×5 matrix A has fewer than 5 distinct eigenvalues, then A is not diagonalizable.

(k) $B = \{1, 1 + x, 1 - x\}$ is a basis for $\mathbb{P}_2$, the vector space of all polynomials with degree less than or equal to 2.

(l) $T(x_1, x_2, x_3) = (x_1, x_2 + 1, x_1, x_3 - x_2)$ gives a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$.

(m) An $n \times n$ matrix is diagonalizable if and only if the sum of the dimensions of its eigenspaces is n.

(n) A matrix that is row equivalent to a diagonal matrix is diagonalizable.

(o) An $n \times n$ matrix that is row equivalent to I_n is diagonalizable.

(p) Nonzero mutually orthogonal vectors can be linearly dependent.

3. Solve the following systems:

$$\begin{cases} x_1 + 3x_2 - 5x_3 = 2 \\ x_1 + x_2 - 8x_3 = 4 \end{cases}$$

4. Find h and k such that the system

$$\begin{array}{ccccc} x & + & 2y & = & k \\ x & + & hy & = & 1 \end{array}$$

has
(a) no solution.
(b) a unique solution.
(c) infinitely many solutions.

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 2x_1 + 3x_2 + 2x_3, 3x_1 + 8x_2 + 2x_3).$$

Find the standard matrix representation of T. Is T invertible? If so, find T^{-1}.

6. Find the determinant of

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 0 & 1 & 2 \\ 1 & 4 & 1 \end{pmatrix}$$

7. Suppose that A is a 3×3 matrix with determinant 5. Find det(2A).

8. Show that the set of vectors of the form

$$\begin{pmatrix} a - b \\ b \\ a \end{pmatrix}, a, b \in \mathbb{R}$$

in $\mathbb{R}^3$ form a subspace of $\mathbb{R}^3$. What is its dimension? Describe this space geometrically.

9. Is the set $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ such that } y = x^2 \right\}$ a subspace of $\mathbb{R}^2$?

10. Consider the vector space $\mathbb{P}_2$, the vector space of all polynomials with degree less than or equal to 2. Determine whether $\{1-3x+2x^2, 1+x+4x^2, 1-7x\}$ is a basis for $\mathbb{P}_2$.

11. Let
$$A = \begin{pmatrix} 3 & 4 & 0 & 7 \\ 1 & -5 & 2 & -2 \\ -1 & 4 & 0 & 3 \\ 1 & -1 & 2 & 2 \end{pmatrix}$$
Find bases for the row space, column space and null space of A.

12. Calculate the determinant of the matrix
$$A = \begin{pmatrix} 7 & 0 & -10 \\ 5 & 2 & -10 \\ 5 & 0 & -8 \end{pmatrix}$$

13. Consider the bases $\mathcal{B} = \left\{ \begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{pmatrix} -4 & 0 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 2 & -5 \end{pmatrix} \right\}$ for the vector space V of lower triangular 2×2 matrices with zero trace.

    (a) Find the matrix of change of coordinates fro the basis from $\mathcal{C}$ to $\mathcal{B}$.

    (b) Find the coordinates of M in the basis $\mathcal{B}$ if the coordinates of M in $\mathcal{C}$ are $[M]_C = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

14. Consider the matrix:
$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$
(a) Find the eigenvalues of A.
(b) Find a diagonal matrix D and an invertible matrix P such that $A = P^{-1}DP$.
(c) Compute A^5.

15. (a) Find an example of $A \in \mathbb{M}_{2\times 2}(\mathbb{R})$ with no real eigenvalues.
(b) Find an example of $A \in \mathbb{M}_{2\times 2}(\mathbb{R})$ with one (repeated) real eigenvalue, whose eigenspace is two-dimensional.
(c) Find an example of $A \in \mathbb{M}_{2\times 2}(\mathbb{R})$ with one (repeated) real eigenvalue, whose eigenspace is one-dimensional.

16. Write the matrix $\begin{pmatrix} 3 & 0 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 2 \end{pmatrix}$ in Jordan form, that is, as PDP^{-1} where D has the form $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$

17. Write the matrix $\begin{pmatrix} -1 & 2 & -2 \\ 0 & 2 & -1 \\ 4 & -3 & 4 \end{pmatrix}$ in Jordan form, that is, as PDP^{-1} where D has the form $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix}$

18. Write the matrix $\begin{pmatrix} -9 & -8 \\ 13 & 11 \end{pmatrix}$ as PDP^{-1} where D has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

19. Use the Gram-Schmidt process to find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors:
$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 4 \\ -2 \end{pmatrix}$$