Section 5.1. Eigenvalues and eigenvectors

Definition: If A is an $n \times n$ matrix, then a vector $\vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$, is an *eigenvector* of A if

$$A\vec{x} = \lambda\vec{x}$$

for some scalar λ, which is called an *eigenvalue* of A, and then $\vec{x}$ is an *eigenvector* corresponding to the eigenvalue λ.

The set of all the eigenvectors of an eigenvalue λ, together with the vector $\vec{0}$, is called the *eigenspace* corresponding to the eigenvalue λ.

How to find eigenvalues and eigenvectors

Note that we have to solve the *eigenvector equation*

$$A\vec{x} = \lambda\vec{x}$$

for both $\vec{x}$ and λ. Since $I \cdot \vec{x} = \vec{x}$, we can rewrite it as

$$A\vec{x} = \lambda I\vec{x}$$

or subtracting $\lambda I \cdot \vec{x}$ from both sides and factoring out $\vec{x}$ on the right, we can write

$$(A - \lambda I)\vec{x} = \vec{0}.$$

This is another form of the eigenvector equation.

To solve it, first observe that the equation means that $\vec{x} \in \mathrm{Null}(A - \lambda I)$, and $\vec{x} \neq \vec{0}$. Therefore, the nullity of $A - \lambda I$ must be at least 1, and this implies

$$\det(A - \lambda I) = 0.$$

This is the *characteristic equation.* The polynomial $\chi_A(\lambda) := \det(A - \lambda I)$ is called the *characteristic polynomial.*

Thus, to find the eigenvalues and eigenvectors of A, we can proceed as follows:

1. Solve the characteristic equation $\det(A - \lambda I) = 0$ for λ.
2. For each λ found in part 1, solve the eigenvector equation $(A - \lambda I)\vec{x} = \vec{0}$ for $\vec{x}$.

For each λ from part 1, the vectors obtained in part 2 will be the eigenvectors corresponding to λ.

Example: Let us find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The characteristic polynomial of A is

$$\chi_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1.$$

Expanding,

$$\chi_A(\lambda) = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0.$$

Thus the eigenvalues are

$$\lambda_1 = 1, \qquad \lambda_2 = 3.$$

Now for each λ, we find the eigenvectors:

Eigenvalue $\lambda_2 = 3$. We have to solve

$$(A - 3I)\vec{x} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It gives the equation $(-1)x + 1y = 0$, or $y = x$. Hence an eigenvector corresponding to the eigenvalue $\lambda = 3$ is

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Eigenvalue $\lambda_1 = 1$.

$$(A - 1I)\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It gives the equation $x + y = 0$, or $y = -x$. Hence an eigenvector corresponding to the eigenvalue $\lambda = 1$ is

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

If A is upper or lower triangular, the eigenvalues are the numbers $a_{11}, a_{22}, \ldots, a_{nn}$ in the diagonal. This is easy to see: If

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}$$

then

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & -2 \\ 0 & 2-\lambda & -2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(3-\lambda) = 0 \quad \Rightarrow \lambda = 1, 2, \text{ or } 3.$$

Exercise: Find the eigenvectors for each λ

Given an eigenvalue λ of a matrix A, the set of eigenvectors corresponding to λ (together with the vector $\vec{0}$) is called the *eigenspace* corresponding to λ.

Note that it is given by the solutions of

$$(A - \lambda I)\vec{x} = \vec{0}$$

or, in other words, by the nullspace of $A - \lambda I$.

Thus, to find a basis for the eigenspace corresponding to λ, we just have to find a basis of

$$\text{Null}(A - \lambda I).$$

Example: Let us find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

The characteristic polynomial of A is

$$\chi_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix}$$

We can find this determinant using Sarrus' rule, for example:

$$\det \begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix} = (2-\lambda)^3 - (2-\lambda) - (2-\lambda) = (2-\lambda)[(2-\lambda)^2 - 2] = (2-\lambda)(\lambda^2 - 4\lambda + 2) = 0$$

The solutions are (using the quadratic formula to solve $(\lambda^2 - 4\lambda + 2) = 0$)

$$\lambda_1 = 2, \qquad \lambda_{2,3} = 2 \pm \sqrt{2}.$$

Now for each λ, we find the corresponding eigenvectors.

Eigenvalue $\lambda_1 = 2$. We solve

$$(A - 2I)\vec{x} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives the equations

$$y = 0, \qquad x + z = 0.$$

Hence an eigenvector is

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Eigenvalue $\lambda_2 = 2 + \sqrt{2}$. We solve

$$(A - \lambda_2 I)\vec{x} = \begin{bmatrix} 2-\lambda_2 & 1 & 0 \\ 1 & 2-\lambda_2 & 1 \\ 0 & 1 & 2-\lambda_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since $2 - \lambda_2 = -\sqrt{2}$, the system becomes

$$-\sqrt{2}\,x + y = 0, \qquad x - \sqrt{2}\,y + z = 0, \qquad y - \sqrt{2}\,z = 0.$$

Solving gives $y = \sqrt{2}\,x$ and $z = x$. Thus an eigenvector is

$$\vec{v}_2 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}.$$

Eigenvalue $\lambda_3 = 2 - \sqrt{2}$. Now $2 - \lambda_3 = \sqrt{2}$, so the system becomes

$$\sqrt{2}\,x + y = 0, \qquad x + \sqrt{2}\,y + z = 0, \qquad y + \sqrt{2}\,z = 0.$$

Solving gives $y = -\sqrt{2}\,x$ and $z = x$. Thus an eigenvector is

$$\vec{v}_3 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}.$$

Exercise: Find the eigenvalues and, for each eigenvalue, a basis for the corresponding eigenspace.

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 0 & 3 \end{bmatrix}.$$

Using the correspondence between linear transformations and matrices, we define:

Definition If $T : V \to V$ is a linear operator, then a nonzero vector $\vec{x} \in V$ is an eigenvector of T if

$$T(\vec{x}) = \lambda\vec{x}$$

for some $\lambda \in \mathbb{R}$. The scalar λ is called an *eigenvalue* of T. The vector $\vec{x}$ is said to be an *eigenvector* corresponding to T. The set of eigenvectors corresponding to an eigenvalue λ is called the *eigenspace* corresponding to λ.