MATH 34 – Differential equations.

Third in-class test. Time allowed: two hours.

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NAME: SOLUTION

Directions: Answer the following questions. The exam will be graded over 100 points; any points you get over 100, up to 110, will count as extra credit. The number to the left of each question is the number of points it is worth.

[25] **1.** Answer the following short questions, justifying your answer:

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a) Suppose that

- $y_1(t)$ and $y_2(t)$ are independent solutions of the differential equation $a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$,
- $y_P(t)$ is a solution of the differential equation $a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$.

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Write down **three more solutions** of the differential equation $a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$.

0 b) Determine if the following statement is true or false, justifying your answer: 'A homogeneous 2×2 linear system of differential equations always has $x(t) = 0$, $y(t) = 0$ as an equilibrium solution'.

5 c) Write down a matrix whose eigenvalues are 0 and 3.

D d) Find a homogeneous linear second degree differential equation with constant coefficients whose general solution is $k_1e^{4t} + k_2e^{-3t}$.

5 e) Determine if the following statement is true or false, justifying your answer, and giving a counterexample if the statement is false: 'Every 2×2 matrix has two different eigenvalues'

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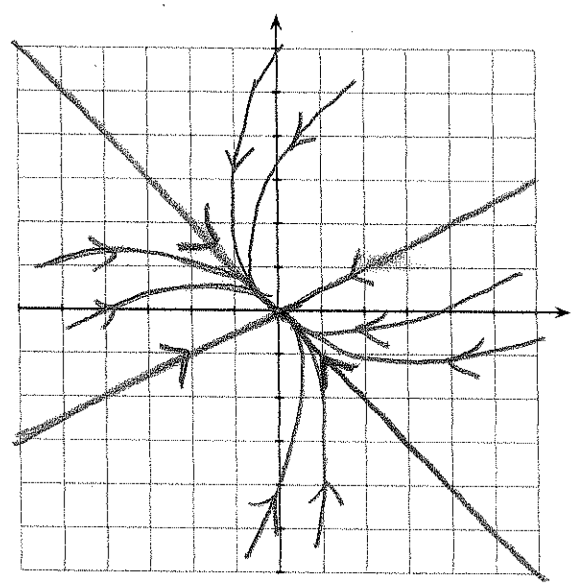
[15] **2.** For the system $\frac{d\vec{Y}}{dt} = \begin{pmatrix} -4 & -2 \\ -1 & -3 \end{pmatrix} \vec{Y}$.

8 a) Find the general solution.

2 b) Determine if the origin is a sink, a source, a saddle, a spiral sink, or a spiral source.

5 c) Sketch the phase plane, drawing carefully all the straight-line solutions (if any) and how the solutions approach (or leave, depending of the case) the origin (if they do). Please use the coordinate axes provided.



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[15] **3.** Solve the initial-value problem $\frac{d\vec{Y}}{dt} = \begin{pmatrix} -5 & 1 \\ -1 & -3 \end{pmatrix} \vec{Y}$, $\vec{Y}(0) = (-1, 2)$.

SOLUTION

① a) By linearity principle (extended)

$$\left.\begin{array}{l} y_1(t) + y_2(t) + y_p(t) \\ 2y_1(t) + y_2(t) + y_p(t) \\ -3y_1(t) + 2y_2(t) + y_p(t) \end{array}\right\} \text{ are all solutions of the nonhomogeneous.}$$

(In general, $k_1 y_1(t) + k_2 y_2(t) + y_p(t)$)

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b) $\boxed{\text{True}}$ A homog linear system has the form

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}; \quad \text{If } x(t)=0,\ y(t)=0,$$

$$LHS = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad RHS = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \text{ So}$$

The constant function $x(t)=0$, $y(t)=0$ is a solution, and therefore $(0,0)$ is an equilibrium solution.

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c) $\begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$; NOTE: $\det\begin{pmatrix} 0-\lambda & 0 \\ 0 & 3-\lambda \end{pmatrix} = -\lambda(3-\lambda) = 0 \Rightarrow \underline{\lambda = 0 \text{ or } \lambda = 3}$.

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d) The exponentials e^{4t} & e^{-3t} say that 4 & -3 are roots of the characteristic polynomial, so it must be $(\lambda-4)(\lambda+3) = \lambda^2 - \lambda - 12$. $\Rightarrow$ Equation is $\boxed{\frac{d^2y}{dt^2} - \frac{dy}{dt} - 12y = 0}$

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e) **FALSE**: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has only $\lambda = 1$ as eigenvalue.

2) a) Find eigenvalues of $\begin{pmatrix} -4 & -2 \\ -1 & -3 \end{pmatrix}$:

$$\det\begin{pmatrix} -4-\lambda & -2 \\ -1 & -3-\lambda \end{pmatrix} = (-4-\lambda)(-3-\lambda)-2 = \lambda^2+7\lambda+10=0$$

$$\Rightarrow \lambda=-2 \text{ or } \lambda=-5.$$

Eigenvectors:

$\lambda=-2$ $\rightarrow$ $\begin{pmatrix} -4-(-2) & -2 \\ -1 & -3-(-2) \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda=-5$ $\rightarrow$ $\begin{pmatrix} -4-(-5) & -2 \\ -1 & -3-(-5) \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$

$$\boxed{\text{Solution: } \vec{Y}(t) = k_1 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

b) As $t\to\infty$, $\vec{Y}(t)\to 0$ (because of the e^{-2t} & e^{-5t}). Thus, $\vec{0}$ is a sink (not a spiral sink since there are no sin or cos involved).

c) In test paper. Note that as $t\to\infty$, e^{-5t} is much smaller than e^{-2t}, so curves approach $(0,0)$ tangent to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

3 Find eigenvalues:

$$\det\begin{pmatrix} -5-\lambda & 1 \\ -1 & -3-\lambda \end{pmatrix} = (-5-\lambda)(-3-\lambda)+1$$

$$= \lambda^2+8\lambda+16 = (\lambda+4)^2=0$$

$$\Rightarrow \boxed{\lambda=-4}$$

One repeated eigenvalue.

Take $\vec{v}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ (initial condition).

Take $\vec{v}_1 = \begin{pmatrix} -5-(-4) & 1 \\ -1 & -3-(-4) \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} =$

$$= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix};$$

Solution:

$$\boxed{\vec{Y}(t) = e^{-4t}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + te^{-4t}\begin{pmatrix} 3 \\ 3 \end{pmatrix}}$$

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4 a) Find eigenvalues: $\det\begin{pmatrix} 2-\lambda & 2 \\ -4 & 6-\lambda \end{pmatrix}$

$$= (2-\lambda)(6-\lambda) + 8$$
$$= \lambda^2 - 8\lambda + 20$$
$$= \lambda^2 - 8\lambda + 16 + 4$$
$$= (\lambda-4)^2 + 4 = 0$$
$$\Rightarrow \underline{\lambda = 4 \pm 2i}$$

Use $\lambda = 4+2i$

$$\begin{pmatrix} 2-(4+2i) & 2 \\ -4 & 6-(4+2i) \end{pmatrix} = \begin{pmatrix} -2-2i & 2 \\ -4 & 2-2i \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$ (for example).

$\Rightarrow$ complex solution is

$$e^{4t}e^{2it}\begin{pmatrix} 1 \\ 1+i \end{pmatrix} = e^{4t}(\cos 2t + i\sin 2t)\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$= \underbrace{e^{4t}\begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix}}_{\text{Real part is a solution}} + i\, \underbrace{e^{4t}\begin{pmatrix} \sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}}_{\text{Imaginary part is a solution.}}$$

$$\Rightarrow \vec{Y}(t) = K_1 e^{4t}\begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} + K_2 e^{4t}\begin{pmatrix} \sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}.$$

b) As $t \to \infty$, $e^{4t} \to \infty$, so $(0,0)$ is a <u>spiral source</u>.

c) In test paper. Note: to find the turning direction, note that at $x=1, y=0$, $\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$. This

$\searrow$ ($\leftarrow$) says it is turning clockwise.

5 Solve homogeneous!

$$\frac{d^2y}{dt^2} + 9y = 0 \ ; \text{ characteristic poly: } \lambda^2 + 9 = 0$$

$$\Rightarrow \lambda^2 = -9 \Rightarrow \lambda = \pm 3i.$$

Solution of homo:

$$k_1 \cos 3t + k_2 \sin 3t.$$

For the non homogeneous, try $y_P = A \sin 6t$

(note: we do not need a term w/ cosine since there is no $\frac{dy}{dt}$ term). Then:

$$\frac{d^2 y_P}{dt^2} = -36 A \sin 6t, \text{ so}$$

$$-36 A \sin 6t + 9 A \sin 6t = 27 \sin 6t \Rightarrow -27A = 27$$

$$\Rightarrow \boxed{A = -1}$$

General solution:

$$y(t) = k_1 \cos 3t + k_2 \sin 3t - \sin 6t,$$

Now, $y'(t) = -3k_1 \sin 3t + 3k_2 \cos 3t - 6 \cos 6t$.

So, $y(0) = k_1 = 0$, $y'(0) = 3k_2 - 6 = 0$

$\Rightarrow k_1 = 0$, $k_2 = 2$. Thus, solution is

$$\boxed{y(t) = 2 \sin 3t - \sin 6t}$$

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a) Characteristic poly: $\lambda^2+4\lambda+13=0$

$$\Rightarrow (\lambda^2+4\lambda+4)+9=0$$
$$\Rightarrow (\lambda+2)^2=-9$$
$$\underline{\lambda=-2\pm 3i}$$

$\Rightarrow$ Solution of homo is

$$\boxed{y_H(t)=k_1 e^{-2t}\cos 3t+k_2 e^{-2t}\sin 3t}$$

b) Complexify & try $y_H = Ae^{2ti}$ as solution. Take real part at the end:

$$\frac{dy_H}{dt}=2iAe^{2ti};\quad \frac{d^2y_H}{dt^2}=-4Ae^{2ti}$$

$$-4Ae^{2ti}+4(2iAe^{2ti})+13Ae^{2ti}=5e^{2ti}$$

$$(9+8i)A=5 \Rightarrow A=\frac{5}{9+8i}=\frac{5(9-8i)}{145}=\frac{9-8i}{29}.$$

$9^2+8^2=145$

Complex solution:

$$\boxed{\frac{9-8i}{29}e^{2ti}}.$$ Now write $\frac{9-8i}{29}$ as $Re^{i\phi}$,

where: $R=\left|\frac{9-8i}{29}\right|=\frac{\sqrt{145}}{25}$, and $\tan\phi=-\frac{8}{9}$

$\Rightarrow \phi\approx -41^\circ \Rightarrow$ complex solution is

$$\frac{\sqrt{145}}{5}e^{-i\cdot 41^\circ}e^{2ti}=\frac{\sqrt{145}}{5}e^{i(2t-41^\circ)}=\frac{\sqrt{145}}{5}\big(\underbrace{\cos(2t-41^\circ)}_{\text{Real part}}+i\sin(2t-41^\circ)\big)$$

Solution: $\boxed{y(t)=k_1e^{-2t}\cos 3t+k_2e^{-2t}\sin 3t+\frac{\sqrt{145}}{5}\cos(2t-41^\circ)}$

c) As $t\to\infty$, $e^{-2t}\to 0$, so $y(t)\sim\frac{\sqrt{145}}{5}\cos(2t-41^\circ)$

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a) $$\boxed{\frac{dW}{dt} = \frac{dx_1}{dt} y_2 + x_1 \frac{dy_2}{dt} - \frac{dx_2}{dt} y_1 - x_2 \frac{dy_1}{dt}}$$

b) $\frac{dx_i}{dt} = a x_i + b y_i$ and $\frac{dy_i}{dt} = c x_i + d y_i$, $i = 1, 2$,

therefore:

$$\frac{dW}{dt} = (a x_1 + \cancel{b y_1}) y_2 + x_1 (\cancel{c x_2} + d y_2) - (a x_2 + \cancel{b y_2}) y_1 - x_2 (\cancel{c x_1} + d y_1)$$

$$= a x_1 y_2 + d x_1 y_2 - a x_2 y_1 - d x_2 y_1$$

$$= (a+d) x_1 y_2 - (a+d) x_2 y_1 = (a+d) \underbrace{(x_1 y_2 - x_2 y_1)}_{W}$$

$$= \underline{\underline{(a+d) W}}$$

c) $$\frac{dW}{W} = (a+d) \Rightarrow \ln W = (a+d)t + c \Rightarrow \boxed{W(t) = Ce^{(a+d)t}}$$

d) Note: $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}, \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$ lin. indep $\Leftrightarrow W(t) \neq 0$

(by definition).

Now: $\begin{pmatrix} x_1(0) \\ y_1(0) \end{pmatrix}, \begin{pmatrix} x_2(0) \\ y_2(0) \end{pmatrix}$ lin indep $\Leftrightarrow W(0) \neq 0$

$\Leftrightarrow W(t) \neq 0$.

But since $W(t) = Ce^{(a+d)t}$, if $W(0) \neq 0$, that means $W(0) = Ce^0 = C \neq 0 \Rightarrow W(t) = Ce^{(a+d)t} \neq 0$

for all t.