SOLUTION Test 2.

①

a) By the linearity principle, $y_p(t) + c\,y_H(t)$ is a solution for any value of c. Thus, for example,

$$\boxed{\begin{array}{l} y_p(t) + y_H(t) \\ y_p(t) + 2y_H(t) \\ y_p(t) + 3y_H(t) \end{array}}$$

are solutions

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b) $\boxed{\text{True}}$ A linear system has the form

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

If we substitute $x=0$, $y=0$, we get $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} = 0$.

Therefore

$$\left.\begin{array}{l} 0 = \dfrac{dx}{dt} = a.0 + b.0 = 0 \\ 0 = \dfrac{dy}{dt} = c.0 + d.0 = 0 \end{array}\right\} \quad x=0 \text{ \& } y=0 \text{ is a solution of the system.}$$

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c) Since $(e^{2t})' = 2e^{2t}$, one such equation is

$$\boxed{y' = 2y}$$

d) Write $v = \frac{dy}{dt}$. Then $\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dv}{dt}$.

The given equation thus reads: $\underbrace{\frac{dv}{dt}}_{\frac{d^2y}{dt^2}} + 4\underbrace{v}_{\frac{dy}{dt}} - 3y = 0$, so

$$\boxed{\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = 3y - 4v \end{cases}}$$

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(2) $\frac{dy}{dt} - \frac{3}{t}y = 2t^3\cos 2t$.

Integrating factor: $\mu = e^{\int -\frac{3}{t}dt} = e^{-3\ln t} = e^{\ln t^{-3}} = t^{-3}$.

Multiply over by μ:

$$t^{-3}\frac{dy}{dt} - 3t^{-2}y = 2\cos 2t$$

$$\frac{d}{dt}\left(t^{-3}y\right) = 2\cos 2t$$

$$t^{-3}y = \sin 2t + C$$

$$\Rightarrow \boxed{y = t^3\sin 2t + Ct^3}$$

③ $\frac{dy}{dt} = -5y + 3e^t$; $y(0) = 2$. Here are <u>two ways</u>!

a) Integrating factor:

$$\frac{dy}{dt} + 5y = 3e^t \longrightarrow \mu = e^{\int 5\,dt} = e^{5t}$$

$$\Rightarrow \quad e^{5t}\frac{dy}{dt} + 5e^{5t}y = 3e^{6t}$$

$$\Rightarrow \int \frac{d}{dt}\left(e^{5t}y\right) = \int 3e^{6t}$$

$$\Rightarrow \quad e^{5t}y = \frac{1}{2}e^{6t} + C$$

$$\Rightarrow \boxed{y = \frac{1}{2}e^{t} + Ce^{-5t}}$$

$$2 = y(0) = \frac{1}{2}e^0 + Ce^0 = \frac{1}{2} + C \Rightarrow \boxed{C = \frac{3}{2}}$$

SOLUTION: $\boxed{y(t) = \frac{1}{2}e^t + \frac{3}{2}e^{-5t}}$

b) Solve $\frac{dy}{dt} = -5y \Rightarrow \boxed{y_H = Ce^{-5t}}$,

Solve $\frac{dy}{dt} = -5y + 3e^t$. Try $y = Ae^t$.

$$\frac{d(Ae^t)}{dt} = Ae^t = -5Ae^t + 3e^t \Rightarrow A = -5A + 3$$

$$\Rightarrow 6A = 3 \Rightarrow A = \frac{1}{2}.$$

$$\Rightarrow \boxed{y_P = \frac{1}{2}e^t}, \text{ and } \boxed{y = \frac{1}{2}e^t + Ce^{-5t}}.$$

As before, $2 = y(0) = \frac{1}{2}e^0 + Ce^0 = \frac{1}{2} + C \Rightarrow C = \frac{3}{2}$

$$\Rightarrow \boxed{y(t) = \frac{1}{2}e^t + \frac{3}{2}e^{-5t}}$$

4 $\begin{cases} \dfrac{dx}{dt} = 2x \\ \dfrac{dy}{dt} = 2x - y \end{cases}$ $\Rightarrow$ $\boxed{x = C_1 e^{2t}}$

$\leadsto \dfrac{dy}{dt} = 2C_1 e^{2t} - y$

$\Rightarrow \dfrac{dy}{dt} + y = 2C_1 e^{2t}$. Int. factor: $\mu = e^{\int 1\,dt} = e^t$

$\Rightarrow e^t \dfrac{dy}{dt} + e^t y = 2C_1 e^{3t}$

$\Rightarrow \int \dfrac{d}{dt}(e^t y) = \int 2C_1 e^{3t}$

$\Rightarrow e^t y = \frac{2}{3} C_1 e^{3t} + C_2$

$y(t) = \frac{2}{3} C_1 e^{2t} + C_2 e^{-t}$

SOLUTION $\boxed{\begin{array}{l} x(t) = C_1 e^{2t} \\ y(t) = \frac{2}{3} C_1 e^{2t} + C_2 e^{-t} \end{array}}$

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5 Equilibrium solutions are of the form $x =$ constant, $y =$ constant, so $\dfrac{dx}{dt} = \dfrac{dy}{dt} = 0$. Thus we need to solve the system of algebraic equations

$$\begin{cases} 0 = x(2x + y - 6) \Rightarrow (\text{either } x = 0 \text{ or } 2x + y - 6 = 0) \\ \qquad\qquad\qquad\qquad \text{AND} \\ 0 = y(-2x + y - 2 \Rightarrow (\text{either } y = 0 \text{ or } -2x + y - 2 = 0) \end{cases}$$

thus, we have 4 cases

1) $\boxed{x=0,\ y=0}$

2) $x=0$ & $-2x+y-2=0 \Rightarrow \underline{x=0}$ & $-2\cdot 0+y-2=0 \Rightarrow \boxed{y=2,\ x=0}$

3) $2x+y-6=0$ & $y=0 \Rightarrow 2x+0-6=0$ & $y=0 \Rightarrow \boxed{x=3,\ y=0}$

4) $2x+y-6=0$ & $-2x+y-2=0$. Solve the system:

$$\begin{cases} 2\cancel{x}+y-6=0 \\ -\cancel{2}x+y-2=0 \end{cases} \qquad 2x+4-6=0 \Rightarrow \boxed{x=1}$$

$$2y-8=0 \Rightarrow \boxed{y=4}$$

Therefore, the equilibrium solution are:

$$\boxed{(0,0),\ (0,2),\ (3,0) \text{ and } (1,4)}$$

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(6) See last page

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(7)

	x	y	$\frac{dx}{dt}$	$\frac{dy}{dt}$	
$t=0$	1	1	-1	1	$\to x_1 = 1-1\cdot 1 = 0,\ y_1 = 1+1\cdot 1 = 2$
$t=1$	0	2	-2	-2	$\to x_2 = 0-2\cdot 1 = -2,\ y_2 = 2-2\cdot 1 = 0$
$t=2$	-2	0	0	-4	$\to x_3 = -2+0\cdot 1 = -2,\ y_3 = 0-4\cdot 1 = -4$
$t=3$	-2	-4	4	0	$\to x_4 = -2+4\cdot 1 = 2;\ y_4 = -4+0\cdot 1 = -4$
$t=4$	2	-4			

Thus, $\boxed{x(4) \approx 2,\ y(4) \approx -4}$

(according to Euler's method w/ $\Delta t = 1$).

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a) In the absence of the other, each species grows without bound.

b) They get along badly: their interaction (the term with "xy") has a negative coefficient, which means that it has a negative effect in the other's growth.

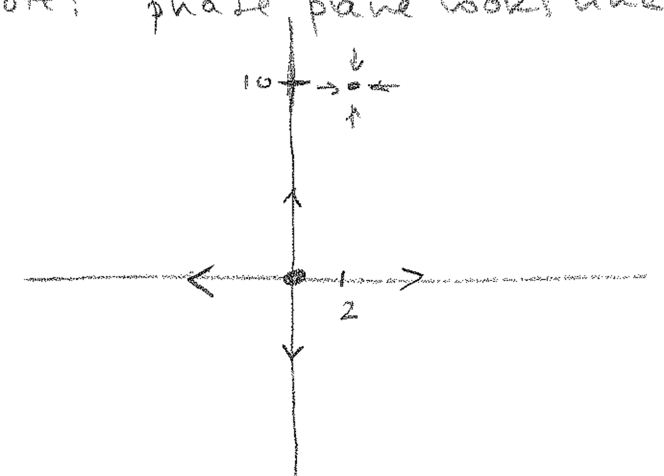
c) Solve: $0 = 10x - xy \Rightarrow 0 = x(10-y)$

$0 = 4y - 2xy \Rightarrow 0 = 2y(2-x)$

$\Rightarrow$ Equilibrium levels are.

$\boxed{(0,0),\ (2,10)}$

(Note: phase plane looks like

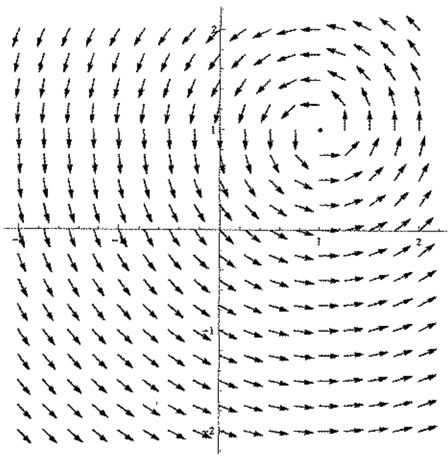


[12] **9.** Eight systems of equation and four direction fields are given below. Determine the system that corresponds to each direction field and state briefly how you know that your choice is correct. 7/8

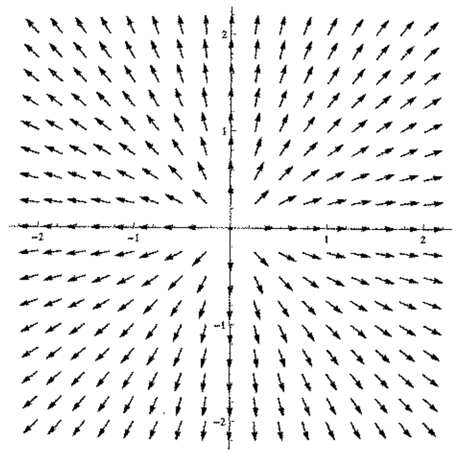
a) $\begin{cases} \frac{dx}{dt} &= x \\ \frac{dy}{dt} &= y \end{cases}$ **b)** $\begin{cases} \frac{dx}{dt} &= -y \\ \frac{dy}{dt} &= x \end{cases}$ **c)** $\begin{cases} \frac{dx}{dt} &= -y+1 \\ \frac{dy}{dt} &= x-1 \end{cases}$

d) $\begin{cases} \frac{dx}{dt} &= \cos(\pi x) \\ \frac{dy}{dt} &= 0 \end{cases}$ **e)** $\begin{cases} \frac{dx}{dt} &= 1 \\ \frac{dy}{dt} &= 3xy \end{cases}$ **f)** $\begin{cases} \frac{dx}{dt} &= x-1 \\ \frac{dy}{dt} &= y^2-1 \end{cases}$

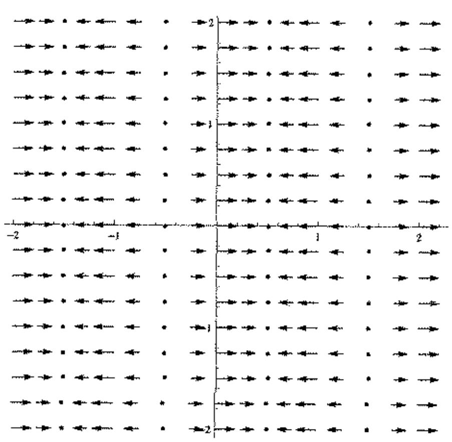
(i)



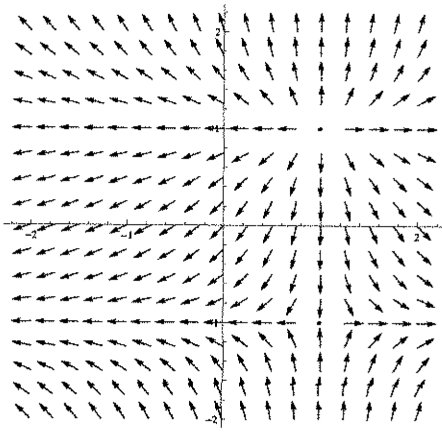
(ii)



(iii)



(iv)



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(i) Has an equilibrium point (i.e. $\frac{dx}{dt} = \frac{dy}{dt} = 0$) at $(1, 1)$. The only one w/ this property is $\boxed{c}$

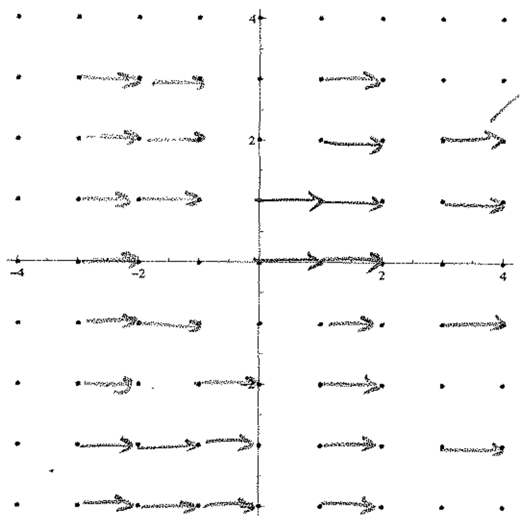
(ii) Has $(0,0)$ as equilibrium point, which eliminates c), e), f). Also, if $y=0$, $\frac{dy}{dt}=0$ (it is horizontal in x-axis) This eliminates b). Thus it is $\boxed{a}$.

(iii) Horizontal always $\Rightarrow \frac{dy}{dt} = 0 \Rightarrow \boxed{d}$

(iv) Has $\frac{dy}{dt}=0$ at $y=1$, $y=-1$. Thus it is $\boxed{f}$

[12] **6.** Draw, in the axes provided below, the vector field and phase portrait for the system 8/8

$$\begin{cases} \dfrac{dx}{dt} = 1 \\ \dfrac{dy}{dt} = 0 \end{cases}$$



always $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

etc.

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[12] **7.** For the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = -y \\ \dfrac{dy}{dt} = 2x - y \\ x(0) = 1 \\ y(0) = 1 \end{cases}$$

use Euler's method with step size $\Delta t = 1$ to approximate $\vec{Y}(4)$.

[NOTE: you can use a calculator here.]

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[12] **8.** Let $x(t)$ and $y(t)$ be the populations (at time t) of two species of animals. The behavior of the two populations is modelled by the system of equations

$$\begin{cases} \dfrac{dx}{dt} = 10x - xy \\ \dfrac{dy}{dt} = 4y - 2xy \end{cases}$$

a) What happens to each species in the absence of the other?

b) How do the two species get along? How do you know?

c) What are the equilibrium population levels?