SOLUTION. MATH 34. TEST 1

① a) Substitute: if $y(t) = \frac{1}{t-1}$, $\frac{dy}{dt} = -\frac{1}{(t-1)^2}$, and

$-y^2 = -\frac{1}{(t-1)^2}$. Therefore, if $y = \frac{1}{t-1}$, then

$\frac{dy}{dt} = -y^2$, so it $\boxed{\text{IS}}$ a solution.

b) As usual, try $y = $ constant. Since $y(0)=0$, try the solution $y(t)=0$. If $y(t)=0$,

$\frac{dy}{dt} = 0$ and $e^y - 1 = e^0 - 1 = 1-1 = 0$

Therefore, $\boxed{y(t)=0}$ is a solution.

c) Again, try $y = $ constant. Since $y(0)=2$, try $y(t)=2$.
If $y(t)=2$, $\underbrace{\frac{dy}{dt}}_{0} + 2\underbrace{y^2}_{2^2} = 0+8 = 8$. ✓ Therefore,
$\boxed{y(t)=2}$ is a solution.

d) An equation $\frac{dy}{dt} = f(t,y)$ is <u>autonomous</u> if f does not depend on t. That means that we can write
$\frac{dy}{dt} = f(y)$. But then we can separate the variables:
$\frac{dy}{f(y)} = dt$. Therefore, the statement is $\boxed{\text{TRUE}}$

② Notes:

Let differential equation whose slope field is given be $\frac{dy}{dt} = f(t,y)$. Then,

- Cannot be autonomous (i.e., f must depend on t) because the slope field changes as t changes. This eliminates a) and f).

- f cannot be independent of y, because the slope changes as y changes. This eliminates e) and d).

We are left with c) and b). Now note that the slopes at $y=-2$ are 0 for all t. Therefore, $f(t,-2)$ must be 0 for all t. This is not the case in c): $\left[\, t\left(\underbrace{(-2)^2}_{4} - \frac{1}{4}\right) = t \cdot \frac{3}{4} \neq 0 \right]$.

However, in b), $\underbrace{((-2)+2)}_{0}((-2)+t) = 0 \cdot ((-2)+t) = 0.$

Therefore, $\boxed{b)}$ has the slope field sought.

3) Solve $\frac{dy}{dt} = y^2 \cos t + \cos t$

$= (y^2+1)\cos t$ (separable).

$\longrightarrow \quad \frac{dy}{y^2+1} = \cos t \, dt$

$\longrightarrow \quad \int \frac{dy}{y^2+1} = \int \cos t \, dt$

$\longrightarrow \quad \arctan y = \sin t + C$

$\longrightarrow \quad y = \tan(\sin t + C)$.

Use initial condition:

$0 = y(0) = \tan(\underbrace{\sin 0}_{=0} + C) = \tan C$

$\Rightarrow C = 0$ (or $C = \pm\pi, \pm 2\pi$, etc; this does not matter)

$\Rightarrow \boxed{y = \tan(\sin t)}$ is the solution.

4) $\frac{dy}{dt} = t^2 y + 2$, $y(0) = 0$. Use Euler w/ $\Delta t = 1$.

$t_0 = 0$, $y_0 = 0$

$t_1 = 1$. ; $y(1) \approx y_1 = y_0 + f(t_0, y_0)\Delta t$

$$= 0 + \underbrace{f(0,0)}_{0^2 \cdot 0 + 2 = 2} \cdot 1 = 0 + 2 \cdot 1 = \underline{2}$$

So $\boxed{y_1 = 2}$

$t_2 = 2$; $y(2) \approx y_2 = y_1 + f(t_1, y_1)\Delta t$

$$= 2 + \underbrace{f(1,2)}_{1^2 \cdot 2 + 2 = 4} \cdot 1 = 2 + 4 \cdot 1 = \underline{6},$$

So $\boxed{y_2 = 6}$

$t_3 = 3$; $y(3) \approx y_3 = y_2 + f(t_2, y_2)\Delta t$

$$= 6 + \underbrace{f(2,6)}_{2^2 \cdot 6 + 2 = 26} \cdot 1 = 6 + 26 \cdot 1 = \underline{32}.$$

Therefore, $\boxed{y(3) \approx 32}$ according to Euler's method w/ step 1.

(ex. 5 cont.)

$$|300-m| = e^{-\frac{t}{75}-C} = e^{-\frac{t}{75}} \cdot \underbrace{e^{-C}}_{K} = K e^{-\frac{t}{75}}.$$

Note that $m(0)=75<300$, and $m=300$ is a solution. Therefore, $m(t)<300$ always, and we can remove the absolute value:

$$300-m = K e^{-\frac{t}{75}}$$

$$\Rightarrow \quad m = 300 - K e^{-\frac{t}{75}}.$$

Now, $m(0)=75 \Rightarrow 75 = 300 - K \cdot 1 \Rightarrow \boxed{K=225}$

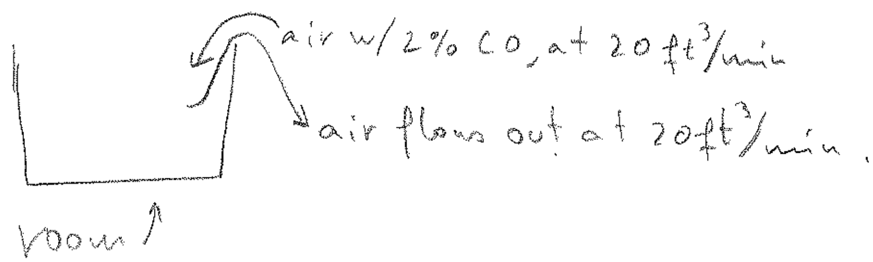
Solution $\boxed{m = 300 - 225\, e^{-\frac{t}{75}}}$.

b) As $t \to \infty$, $e^{-\frac{t}{75}} \to 0$, so $m \to 300$.

thus, after a long time there will be $300\ ft^3$ of CO in the room.

c) $m(10) = 300 - 225\, e^{-\frac{10}{75}} = \boxed{103.09\ ft^3}$

(5)



(30×5×10 = 1500 ft³)

Let m = amount (volume) of CO, in ft³.

$\frac{dm}{dt}$ = volume of CO in/min − volume of CO out/min.

<u>Volume in</u>: 2% of 20 ft³/min, or $0.02 \cdot 20$ ft³/min
$= \underline{0.4 \text{ ft}^3/\text{min}}$

<u>Volume out</u>: Let V be the volume of CO that goes out dissolved in the mixed air, per minute.

There are: $\left\{ \begin{array}{l} m \text{ ft}^3 \text{ in a room of volume } 1500 \text{ ft}^3, \\ V \text{ ft}^3 \text{ in a volume of } 20 \text{ ft}^3 \end{array} \right.$

$\Rightarrow \quad \frac{m}{V} = \frac{1500}{20} \quad \Rightarrow \quad \underline{V = \frac{m}{75}} \text{ ft}^3/\text{min}$

The initial value problem is

$$\boxed{\frac{dm}{dt} = 0.4 - \frac{m}{75}, \quad m(0) = 0.05 \cdot 1500 = 75.}$$

Let us solve it: $\frac{dm}{dt} = 0.4 - \frac{m}{75} = \frac{300-m}{75}$

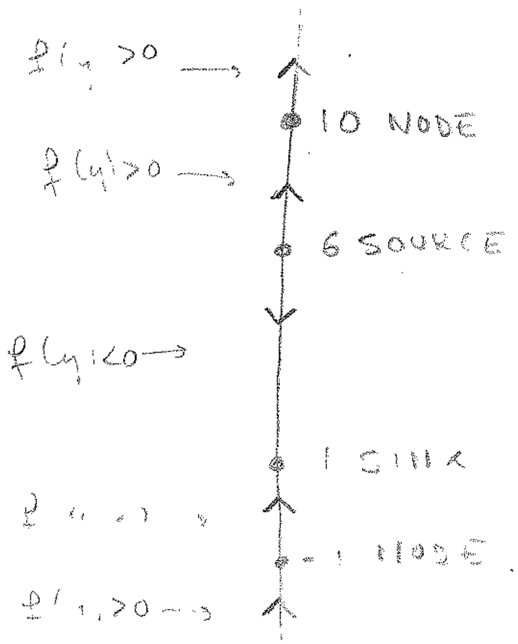
$\Rightarrow \quad \frac{1}{300-m}\, dm = \frac{dt}{75} \quad \Rightarrow \quad -\ln|300-m| = \frac{t}{75} + C \quad \Rightarrow$

(OVER)

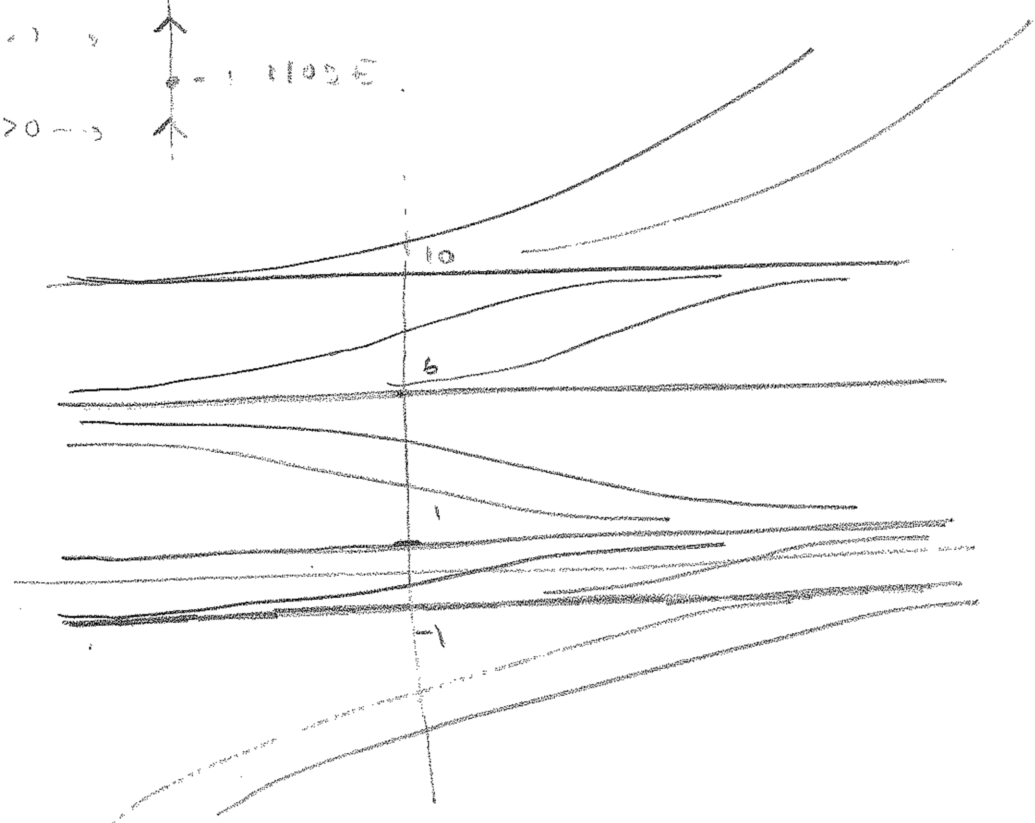
⑥ $$\frac{dy}{dt} = (y+1)^2 (y-1)(y-6)^3 (y-10)^2 = f(y)$$

a) this is 0 at $-1, 1, 6, 10$.

eq. pts.



b)



(7) a) If $y_1(t)$ is a solution, then $\frac{dy_1}{dt} = f(y_1)$.

If $y_1(t)$ has a maximum at $t = t_0$, then $\frac{dy_1}{dt}\Big|_{t=t_0} = 0$ (maximums are critical points).

Therefore, at $t = t_0$, $\underline{\underline{0}} = \frac{dy_1}{dt}\Big|_{t=t_0} = f(\underbrace{y_1(t_0)}_{y_0})$

$= \underline{\underline{f(y_0)}}$

Hence, $\underline{f(y_0) = 0}$.

b) $y_2(t) = y_0$ is a solution because, since y_0 is a constant, $\frac{dy_2}{dt} = \frac{dy_0}{dt} = 0$, and from a) we know $f(y_0) = 0$ also, so $\frac{dy_2}{dt} = 0 = f(y_2)$.

c) y_1 and y_2 are solutions, and $y_1(t_1) = y_0$ and $y_2(t_1) = y_0$ also. Therefore, by uniqueness, $y_1(t) = y_2(t) = y_0$ (we can use the uniqueness theorem because, by hypothesis, f is differentiable).