MATH 34 – Differential equations.

Third in-class test. Time allowed: two hours.

Instructor: Luis Fernández

NAME:____

Directions: Answer the following questions. The exam will be graded over 100 points; any points you get over 100, up to 110, will count as extra credit. The number to the left of each question is the number of points it is worth.

[25] **1.** Answer the following short questions, justifying your answer:

a) Suppose that

- $y_1(t)$ and $y_2(t)$ are independent solutions of the differential equation $a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$,
- $y_P(t)$ is a solution of the differential equation $a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$.

Write down **three more solutions** of the differential equation $a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$.

b) Determine if the following statement is true or false, justifying your answer: 'A homogeneous 2×2 linear system of differential equations always has $x(t) = 0$, $y(t) = 0$ as an equilibrium solution'.

c) Write down a matrix whose eigenvalues are 0 and 3.

d) Find a homogeneous linear second degree differential equation with constant coefficients whose general solution is $k_1 e^{4t} + k_2 e^{-3t}$.

e) Determine if the following statement is true or false, justifying your answer, and giving a counterexample if the statement is false: 'Every 2×2 matrix has two different eigenvalues'

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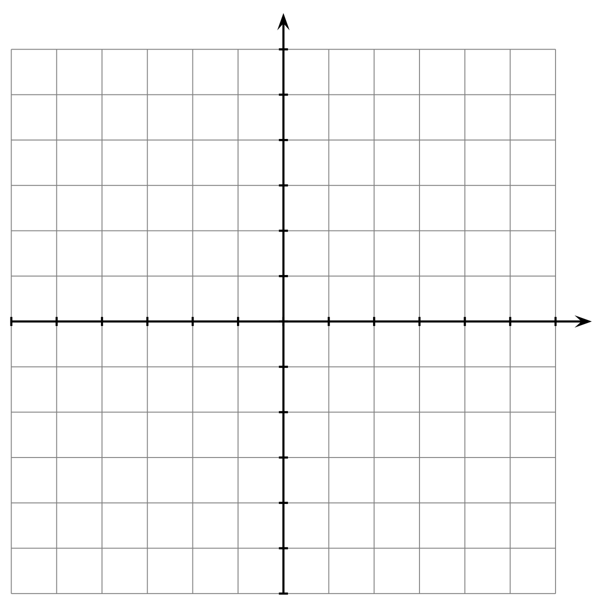
[15] **2.** For the system $\dfrac{d\vec{Y}}{dt} = \begin{pmatrix} -4 & -2 \\ -1 & -3 \end{pmatrix} \vec{Y}$.

a) Find the general solution.

b) Determine if the origin is a sink, a source, a saddle, a spiral sink, or a spiral source.

c) Sketch the phase plane, drawing carefully all the straight-line solutions (if any) and how the solutions approach (or leave, depending of the case) the origin (if they do). Please use the coordinate axes provided.



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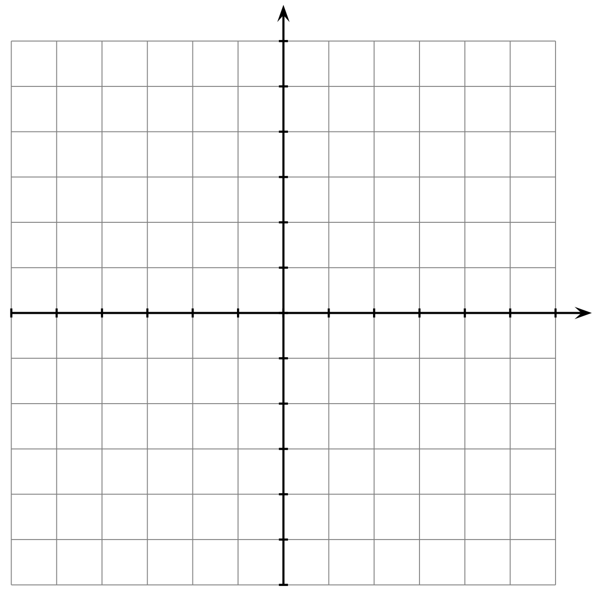
[15] **3.** Solve the initial-value problem $\dfrac{d\vec{Y}}{dt} = \begin{pmatrix} -5 & 1 \\ -1 & -3 \end{pmatrix} \vec{Y}$, $\vec{Y}(0) = (-1, 2)$.

[15] **4.** For the system $\dfrac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix} \vec{Y}$.

a) Find the general solution.

b) Determine if the origin is a sink, a source, a saddle, a spiral sink, or a spiral source.

c) Sketch the phase plane, drawing carefully all the straight-line solutions (if any) and how the solutions approach (or leave, depending of the case) the origin (if they do). Please use the coordinate axes provided.



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[15] **5.** Solve the initial value problem $\dfrac{d^2y}{dt^2} + 9y = 27\sin 6t,\;\; y(0) = 0,\, y'(0) = 0$

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[15] **6.** Consider the nonhomogeneous linear equation $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 13y = 5\cos 2t,$

a) Find the general solution of the associated homogeneous equation $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 13y = 0.$

b) Find a particular solution of the original nonhomogeneous equation and write it in the form $A\cos(t-\phi)$ (or $A\cos(t+\phi)$), where ϕ is the phase given in degrees. Then write the general solution of the original equation.

c) Briefly describe the long-term behavior of the solutions (i.e. the steady state solution).

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[15] **7.** [BONUS] The **Wronskian** of two vector functions $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ is defined by

$$W(t) := x_1(t)y_2(t) - x_2(t)y_1(t).$$

a) Compute $\dfrac{dW}{dt}$ in terms of x_1, x_2, y_1 and y_2 and their derivatives.

b) Suppose that $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are solutions of the system $\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$

Show that $\dfrac{dW}{dt} = (a+d)W(t)$.

c) Find the general solution of the equation $dW/dt = (a+d)W(t)$.

d) Use the previous items to show that if $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are solutions of a linear system, and $(x_1(0), y_1(0))$ and $(x_2(0), y_2(0))$ are linearly independent vectors then, for any value of t, $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are also linearly independent vectors.

[Recall that two vectors (u_1, v_1) and (u_2, v_2) are linearly independent if $u_1v_2 - u_2v_1 \neq 0$.]