Print Name: SOLUTION

Spring 2018 MTH 32 Test 3

Directions: Write your answers in the provided space. **Show all work.**

1. (9 pts) Evaluate the following integral: $\displaystyle\int \frac{x^2+3x-4}{x-2}\,dx$

Divide:

$$\begin{array}{r} x+5 \\ x-2\,\overline{\big)\,\cancel{x^2}+3x-4} \\ \underline{-\cancel{x^2}+2x} \\ 5\cancel{x}-4 \\ \underline{-\cancel{5x}+10} \\ 6 \end{array} \Rightarrow \frac{x^2+3x-4}{x-2} = x+5+\frac{6}{x-2}$$

$$\int \frac{x^2+3x-4}{x-2}\,dx = \int \left(x+5+\frac{6}{x-2}\right)dx$$

$$= \boxed{\frac{x^2}{2} + 5x + 6\,\mathrm{Ln}|x-2| + C}$$

2. (9 pts) Find the exact value of the following integral: $\displaystyle\int_{-\pi}^{\pi} \sin^3 x\cos^2 x\,dx = 0$

Because $\sin^3 x\cos^2 x$ is odd, and $[-\pi,\pi]$ symmetric about 0.

Another (long) way!

$$\int_{-\pi}^{\pi} \sin^3 x\cos^2 x\,dx = \int_{-\pi}^{\pi} \underbrace{\sin^2 x}_{(1-\cos^2 x)}\cos^2 x\sin x\,dx$$

$$= -\int_{x=-\pi}^{x=\pi} (1-t^2)t^2\,dt = \int_{x=-\pi}^{x=\pi} (t^4-t^2)\,dt = \left.\frac{t^5}{5}-\frac{t^3}{3}\right]_{x=-\pi}^{x=\pi}$$

$\cos x = t$

$-\sin x\,dx = dt$

$$= \left.\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3}\right]_{-\pi}^{\pi} = \left(\frac{(-1)^5}{5} - \frac{(-1)^3}{5}\right) - \left(\frac{(-1)^5}{5} - \frac{(-1)^3}{5}\right)$$

$$= \boxed{0}$$

3. (9 pts) Evaluate the following integral: $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$

<u>Method 1</u> put $t = 1+x^2$ (so $x^2 = t-1$)

$dt = 2x\,dx$

$$\int \frac{x^3}{\sqrt{1+x^2}}dx = \frac{1}{2}\int \frac{x^2}{\sqrt{1+x^2}}\cdot 2x\,dx = \frac{1}{2}\int \frac{t-1}{\sqrt{t}}\,dt = \frac{1}{2}\int \left(t^{1/2} - t^{-1/2}\right)dt$$

$$= \frac{1}{2}\left(\frac{2t^{3/2}}{3} - 2t^{1/2}\right) = \boxed{\frac{\sqrt{(x^2+1)^3}}{3} - \sqrt{x^2+1} + C}$$

<u>Method 2</u> put $x = \tan t$, so $dx = \sec^2 t\,dt$

$$\int \frac{x^3}{\sqrt{1+x^2}}dx = \int \frac{\tan^3 t \sec^2 t}{\sqrt{\underbrace{1+\tan^2 t}_{=\sec^2 t}}}dt = \int \frac{\tan^3 t \sec^{\cancel{2}} t}{\cancel{\sec t}}dt$$

$$= \int \underbrace{\tan^2 t}_{1-\sec^2 t} \tan t \sec t\,dt = \int (\sec^2 t - 1)\tan t \sec t\,dt =$$

$\sec t = \sqrt{1+\tan^2 t} = \sqrt{1+x^2}$

$u = \sec t$

$du = \tan t \sec t\,dt$

$$= \int (u^2-1)\,du = -u + \frac{u^3}{3} = -\sec t + \frac{\sec^3 t}{3} = \boxed{-\sqrt{1+x^2} + \frac{\sqrt{(1+x^2)^3}}{3}}$$

4. (9 pts) Evaluate the following integral: $\int_0^1 x\tan^{-1}x\,dx$

by parts: $u = \tan^{-1}x$, $du = \frac{1}{1+x^2}$, $dv = x\,dx$, $v = \frac{x^2}{2}$

$$\int_0^1 x\tan^{-1}x\,dx = \frac{x^2}{2}\tan^{-1}x\Big]_0^1 - \int_0^1 \frac{x^2}{2}\cdot\frac{1}{1+x^2}dx$$

divide: $1+x^2 \,\overline{)\,x^2}$, $\frac{1}{}$; $\underline{x^2+1}$; -1

$$\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$$

$$= \frac{1}{2}\underbrace{\tan^{-1}1}_{\pi/4} - \frac{0}{2}\underbrace{\tan^{-1}0}_{0} - \frac{1}{2}\int_0^1\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\left(x - \tan^{-1}x\right)\Big]_0^1$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[\left(1 - \underbrace{\tan^{-1}1}_{\pi/4}\right) - \left(0 - \underbrace{\tan^{-1}0}_{0}\right)\right] = \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8}$$

$$= \boxed{\frac{\pi}{4} - \frac{1}{2}}$$

5. (9 pts) Evaluate the following integral: $\displaystyle\int \frac{x^2+x+4}{x^3-4x^2+4x}\,dx$

$$x^3-4x^2+4x = x(x^2-2x+4) = x(x-2)^2$$

$$\frac{x^2+x+4}{x^3-4x^2+4x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

$$\Rightarrow \quad x^2+x+4 = A(x-2)^2 + Bx(x-2) + Cx$$

Put $x=0$, get

$$4 = 4A \Rightarrow \boxed{A=1}$$

put $x=2$, get

$$2^2+2+4 = 2C \Rightarrow 2C=10 \Rightarrow \boxed{C=5}$$

The term of degree 2 is

$$x^2 = Ax^2 + Bx^2 \Rightarrow 1 = 1+B \Rightarrow \boxed{B=0}$$

Thus,

$$\int \frac{x^2+x+4}{x^3-4x^2+4x}\,dx = \int\left(\frac{1}{x} + \frac{5}{(x-2)^2}\right)dx$$

$$= \boxed{\ln|x| - \frac{5}{x-2} + C}$$

6. (9 pts) Determine if the integral converges or diverges. If it converges, find its value.

$$\int_1^\infty \frac{dx}{x^2+x}$$

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \Rightarrow 1 = A(x+1) + Bx$$

$$= (A+B)x + A$$

$$\Rightarrow A = 1,\ B = -1$$

$$= \frac{1}{x} - \frac{1}{x+1}$$

$$\int_1^\infty \frac{dx}{x^2+x} = \lim_{b\to\infty} \int_1^b \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \lim_{b\to\infty} \left(\mathrm{Ln}\,x - \mathrm{Ln}(x+1)\right)\Big]_1^b$$

$$= \lim_{b\to\infty} \mathrm{Ln}\frac{x}{x+1}\Big]_1^b = \lim_{b\to\infty} \mathrm{Ln}\frac{b}{b+1} - \mathrm{Ln}\frac{1}{2}$$

$\frac{b}{b+1} \to 1$ as $b \to \infty$

Thus, $\mathrm{Ln}\frac{b}{b+1} \xrightarrow{b\to\infty} 0$

$$= -\mathrm{Ln}\frac{1}{2}$$

$$= \boxed{\mathrm{Ln}\,2}$$

7. (9 pts) Determine if the integral converges or diverges. If it converges, find its value.

$$\int_0^1 \mathrm{Ln}\,x\,dx = \lim_{b\to 0^+} \int_b^1 \mathrm{Ln}\,x\,dx = \lim_{b\to\infty} \left(x\,\mathrm{Ln}\,x\Big]_b^1 - \int_b^1 \frac{x}{x}\,dx\right)$$

$u = \mathrm{Ln}\,x \quad dv = dx$

$du = \frac{1}{x} \quad v = x$

$$= \lim_{b\to 0^+} \left(x\,\mathrm{Ln}\,x - x\Big]_b^1\right) = (1\,\overset{0}{\mathrm{Ln}\,1} - 1) - \lim_{b\to 0^+} (b\,\mathrm{Ln}\,b - b)$$

$$= -1 - \underbrace{\lim_{b\to 0^+} b\,\mathrm{Ln}\,b}_{\to 0} + \lim_{b\to 0^+} b = 0$$

Now: $\lim_{b\to 0^+} b\,\mathrm{Ln}\,b = \lim_{b\to 0^+} \frac{\mathrm{Ln}\,b}{1/b} \overset{L.H}{=} \lim_{b\to 0^+} \frac{1/b}{-1/b^2} = \lim_{b\to 0} -b = 0$

Therefore $\boxed{\int_0^1 \mathrm{Ln}\,x\,dx = -1}$

8. (9 pts) Determine if the integral converges or diverges. If it converges, find its value.

$$\int_1^\infty \left(\tan^{-1} x - \frac{\pi}{2} + \frac{1}{x}\right) dx$$

Let us do $\int \tan^{-1} x \, dx$ first:

$$\int \tan^{-1} x \, dx = x\tan^{-1} x - \int \frac{x\,dx}{1+x^2} = x\tan^{-1}x - \frac{1}{2}\int \frac{dt}{t} =$$

$u = \tan^{-1} x$ $\quad dv = dx$
$du = \frac{dx}{1+x^2}$ $\quad v = x$

$1+x^2 = t$
$2x\,dx = dt$
$x\,dx = \frac{dt}{2}$

$$= x\tan^{-1}x - \frac{1}{2}\ln t = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)$$
$$= x\tan^{-1}x - \ln\sqrt{1+x^2}.$$

Thus,

$$\int_1^\infty \left(\tan^{-1}x - \frac{\pi}{2} + \frac{1}{x}\right)dx = \lim_{b\to\infty}\int_1^b \left(\tan^{-1}x - \frac{\pi}{2} + \frac{1}{x}\right)dx$$
$$= \lim_{b\to\infty}\left(x\tan^{-1}x - \ln\sqrt{1+x^2} - \frac{\pi x}{2} + \ln x\right]_1^b$$

$\ln x - \ln\sqrt{1+x^2} = \ln\frac{x}{\sqrt{1+x^2}}.$

$$= \lim_{b\to\infty}\left(x\left(\tan^{-1}x - \frac{\pi}{2}\right) + \ln\frac{x}{\sqrt{1+x^2}}\right]_1^b$$
$$= \lim_{b\to\infty}\left(b\left(\tan^{-1}b - \frac{\pi}{2}\right) + \ln\frac{b}{\sqrt{1+b^2}}\right) - \left(1\left(\tan^{-1}1 - \frac{\pi}{2}\right) + \ln\frac{1}{\sqrt{2}}\right) = *$$

$\tan^{-1}1 = \pi/4$

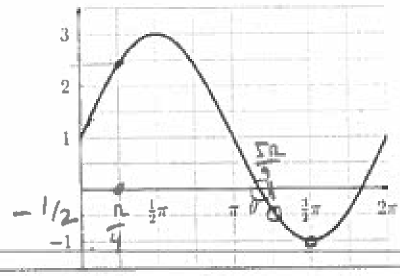
Now, $\lim_{b\to\infty} b\left(\tan^{-1}b - \pi/2\right) = \lim_{b\to\infty}\frac{\tan^{-1}b - \pi/2}{1/b} \overset{LH}{=} \lim_{b\to\infty}\frac{\frac{1}{1+b^2}}{-1/b^2} = -1.$

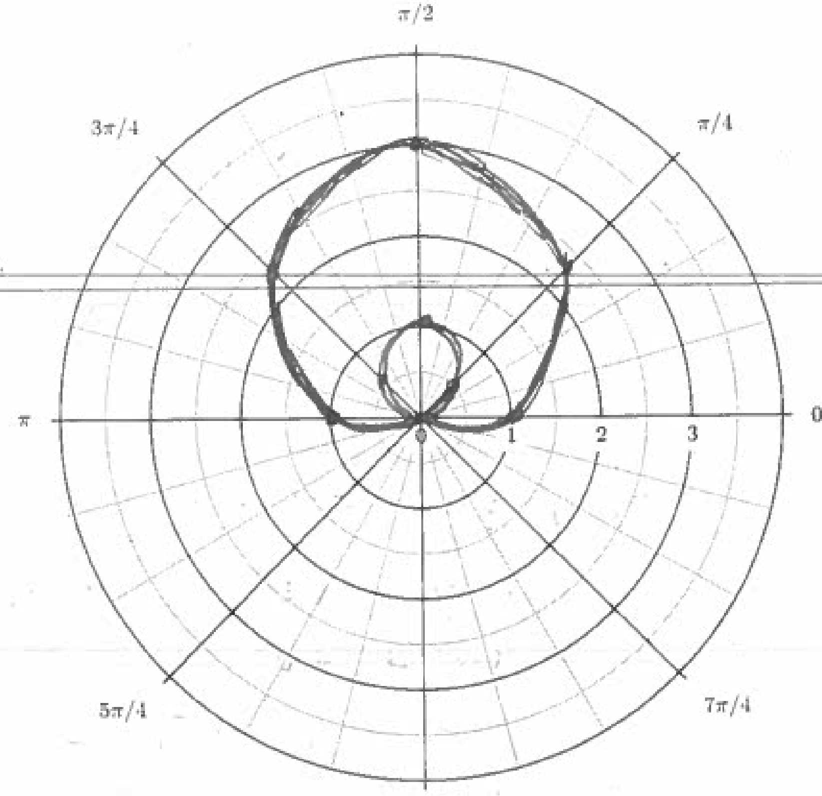
$\lim_{b\to\infty}\ln\frac{b}{\sqrt{1+b^2}} = \ln 1 = 0$

Finally

$$* = -1 + 0 - \left(\left(\frac{\pi}{4} - \frac{\pi}{2}\right) + \ln\frac{1}{\sqrt{2}}\right)$$
$$= \boxed{-1 + \frac{\pi}{4} - \ln\sqrt{2}}$$

9. (9 pts) Sketch the polar graph of $r = 1 + 2\sin(\theta)$ in the axes provided. To make it easier, the cartesian graph is given below.





10. (9 pts) Find the length of the *catenary*, which is the curve with equation $y = a\cosh(x/a)$, where a is a positive constant, between $x = -c$ and $x = c$.

[The catenary is the shape that a chain takes when it hangs from two ends at the same height - *catena* means *chain* in Latin.]

$$\text{Length} = \int_{-c}^{c} \sqrt{1 + \left((a\cosh x/a)'\right)^2}\, dx = \int_{-c}^{c} \sqrt{1 + \left(\sinh \tfrac{x}{a}\right)^2}\, dx$$

$$= \int_{-c}^{c} \sqrt{\cosh^2 \tfrac{x}{a}}\, dx = \int_{-c}^{c} \cosh \tfrac{x}{a}\, dx = a\sinh \tfrac{x}{a}\Big]_{-c}^{c}$$

$$= a\sinh\left(\frac{c}{a}\right) - a\sinh\left(\frac{-c}{a}\right)$$

$$= \boxed{2a\sinh \frac{c}{a}}$$

11. (10 pts) Evaluate the integral: $\int \frac{5x^2+13x+15}{x^3+4x^2+5x}\,dx$.

$$x^3+4x^2+5x = x(x^2+4x+5)$$
$$= x(x^2+4x+4+1)$$
$$= x((x+2)^2+1)$$

Write as

$$\frac{5x^2+13x+15}{x(x^2+4x+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+4x+5}.$$

Get: $5x^2+13x+15 = A(x^2+4x+5)+x(Bx+C)$

Put $x=0$, get $15 = 5A \Rightarrow \boxed{A=3}$

Coefficients of degree 2:

$$5x^2 = Ax^2+Bx^2 = 3x^2+Bx^2 \Rightarrow \boxed{B=2}$$

Coefficients of degree 1:

$$13x = 4\underset{3}{\underset{\shortparallel}{A}}x + Cx \Rightarrow 13 = 12 + C \Rightarrow C = 1.$$

Thus,

$$\int \frac{5x^2+13x+15}{x^3+4x^2+5x}\,dx = \int \frac{3}{x}\,dx + \int \frac{2x+1}{x^2+4x+5}\,dx$$

$t = x^2+4x+5$
$dt = (2x+4)\,dx$

$$= \int \frac{3}{x}\,dx + \int \frac{2x+4}{x^2+4x+5}\,dx - \int \frac{3}{x^2+4x+5}\,dx$$

$$= 3\ln x + \int \frac{1}{t}\,dt - \int \frac{3}{(x+2)^2+1}\,dx$$

$$= \boxed{3\ln x + \ln(x^2+4x+5) - 3\tan^{-1}(x+2) + C}$$

12. (10 pts) The *Gamma* function is defined by

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx.$$

It is very important in many branches of math and in statistics (it models lifespan, for example).

(a) Prove that $\Gamma(p) = (p-1)\cdot\Gamma(p-1)$. $(p>0)$

(b) Find $\Gamma(1)$.

~~(c) Prove that $\Gamma(n) = (n-1)!$ when n is a positive integer. [HINT: use (a) and (b).]~~

(d) Use the identity $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ to prove that $\Gamma(1/2) = \sqrt{\pi}$.

(a) $\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx = -x^{p-1}e^{-x}\Big]_0^\infty + \int_0^\infty (p-1)x^{p-2}e^{-x}\,dx$

$$\boxed{\begin{array}{ll} u = x^{p+1} & dv = e^{-x}dx \\ du = (p-1)\,x^{p-2} & v = -e^{-x} \end{array}}$$

$= \lim_{b\to\infty} -x^{p-1}e^{-x}\Big]_0^b + (p-1)\boxed{\int_0^\infty x^{p-2}e^{-x}\,dx} = \Gamma(p-1)$

$= 0 + (p-1)\,\Gamma(p-1)$

since $\lim_{b\to\infty} b^{p-1}e^{-b} = 0$.

(b) $\Gamma(1) = \int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big]_0^b = \lim_{b\to\infty} -e^{-b} + e^{-0} = 1$

$\Rightarrow \Gamma(1) = 1$

(c) $\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\,\Gamma(n-2) = \dots$

$= (n-1)(n-2)(n-3)\Gamma(n-4) = (n-1)(n-2)\dots 2\cdot\Gamma(1) = \underline{(n-1)!}$

(d) $\Gamma(1/2) = \int_0^\infty x^{-1/2}e^{-x} = 2\int_0^\infty e^{-t^2}dt = 2\,\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$

$$\boxed{\begin{array}{l} x^{1/2} = t \to x = t^2 \\ \frac{1}{2}x^{-1/2}dx = dt \\ x^{-1/2}dx = 2dt \end{array}}$$

Since $\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-t^2}dt = \int_0^\infty e^{-t^2}dt + \int_{-\infty}^0 e^{-t^2}dt$ these are equal (e^{-t^2} is even)

$\Rightarrow \int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$

100 points total + 10 bonus