Print Name: SOLUTION

Spring 2018 MTH 32 Test 2

Directions: Write your answers in the provided space. **Show all work.**

1. (12 pts) Find the derivative of the following functions. Simplify when possible.

(a) $\tan^{-1}(\cosh x)$

$$\boxed{\frac{\sinh x}{\cosh^2 x+1}}$$

(b) $\sinh(4\sin^{-1} x+2)$

$$\boxed{\frac{4\cosh(4\sin^{-1}x+2)}{\sqrt{1-x^2}}}$$

(c) $\sin^{-1} x+\cos^{-1} x = \frac{\pi}{2}$, so its derivative $=\boxed{0}$.

Alternatively,

$$(\sin^{-1}x+\cos^{-1}x)' = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = \boxed{0}$$

(d) $\sinh^{-1} x$ If you remember formula: $\boxed{\frac{1}{\sqrt{x^2+1}}}$

Else: write $y=\sinh^{-1}x$,

$\Rightarrow \sinh y = x \Rightarrow \cosh y \cdot y' = 1$

$$\Rightarrow y' = \frac{1}{\cosh(\sinh^{-1}x)} = \boxed{\frac{1}{\sqrt{x^2+1}}}$$

$\cosh y = \sqrt{\sinh^2 y+1}$

$\sinh^2 y = x^2$

2. (12 pts) Find the exact value of

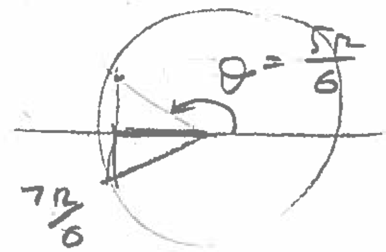
(a) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \boxed{-\frac{\pi}{3}}$

(b) $\sinh 2 = \boxed{\frac{e^2 - e^{-2}}{2}}$

(c) $\sinh^{-1}\left(\frac{e^2 - e^{-2}}{2}\right) = \boxed{2}$, because

(d) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ = angle $\theta \in [0, \pi]$ s.t. $\cos\theta = \cos\frac{7\pi}{6}$

$= \boxed{\frac{5\pi}{6}}$



3. (10 pts) Prove the identity: $\cosh(2x) = \cosh^2 x + \sinh^2 x$.

$$\cosh^2 x + \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + \cancel{2e^x e^{-x}} + e^{-2x}}{4} + \frac{e^{2x} - \cancel{2e^x e^{-x}} + e^{-2x}}{4}$$

$$= \frac{2(e^{2x} + e^{-2x})}{4} = \frac{e^{2x} + e^{-2x}}{2} = \underline{\underline{\cosh(2x)}}$$

4. (20 pts) Evaluate the following limits, indicating the argument carefully.

(a) $\lim\limits_{x\to 0} \dfrac{\cosh x - 1}{x} \overset{LH}{=} \lim\limits_{x\to 0} \dfrac{\sinh x}{1} = \boxed{0}$

(b) $\lim\limits_{x\to\infty} e^{-x}\cosh x = \lim\limits_{x\to\infty} e^{-x}\dfrac{e^{x}+e^{-x}}{2} = \lim\limits_{x\to\infty} \dfrac{1+e^{-2x}}{2}$

$= \boxed{\dfrac{1}{2}}$

(c) $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right) = \lim\limits_{x\to 0^+} \dfrac{e^x-1-x}{x(e^x-1)} \overset{LH}{=} \lim\limits_{x\to 0^+} \dfrac{e^x-1}{e^x-1+xe^x} =$

$\overset{LH}{=} \lim\limits_{x\to 0^+} \dfrac{e^x}{e^x+e^x+xe^x} = \boxed{\dfrac{1}{2}}$

(d) $\lim\limits_{x\to 0}(1-2x)^{1/x} = \lim\limits_{x\to 0} e^{\ln(1-2x)^{1/x}} = \lim\limits_{x\to 0} e^{\frac{1}{x}\ln(1-2x)} = \boxed{e^{-2}}$

Do $\lim\limits_{x\to 0} \dfrac{\ln(1-2x)}{x} \overset{LH}{=} \lim\limits_{x\to 0} \dfrac{\frac{-2}{1-2x}}{1} = -2$

(e) $\lim\limits_{x\to 1^+} x^{1/(x-1)} = \lim\limits_{x\to 1^+} e^{\ln x^{\frac{1}{x-1}}} = \lim\limits_{x\to 1^+} e^{\frac{\ln x}{x-1}} = e^1 = \boxed{e}$

Do $\lim\limits_{x\to 1^+} \dfrac{\ln x}{x-1} = \lim\limits_{x\to 1^+} \dfrac{\frac{1}{x}}{1} = 1$

5. (16 pts) Evaluate ~~the~~ t^2 following integrals.

(a) $\displaystyle\int_0^{1/2} \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}}\,dx$ [Hint: What is $(\sin^{-1} x)'$?]

$t = \sin^{-1} x$

$dt = \frac{1}{\sqrt{1-x^2}}\,dx$

~~If $x = \frac{1}{2}$, $t = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$~~

If $x = 0$, $t = \sin^{-1} 0 = 0$

$$= \int_0^{\pi/6} t^2\,dt = \frac{t^3}{3}\Big]_0^{\pi/6} = \boxed{\frac{\pi^3}{3\cdot 6^3}} = \boxed{\frac{\pi^3}{648}}$$

(b) $\displaystyle\int \frac{2x}{1+x^4}\,dx$ [Hint: What is $(x^2)'$?]

$t = x^2$

$dt = 2x\,dx$

$$= \int \frac{dt}{1+t^2} = \tan^{-1} t + C$$
$$= \boxed{\tan^{-1} x^2 + C.}$$

6. (16 pts) Evaluate the following integrals.

(a) $\displaystyle\int \underset{u}{x}\,\underset{dv}{\sinh x\,dx} = \underset{u}{x}\,\underset{v}{\cosh x} - \int \underset{v}{\cosh x}\,\underset{du}{dx}$

By parts:

$u = x \qquad v = \cosh x$

$du = dx \qquad dv = \sinh x\,dx$

$$= \boxed{x\cosh x - \sinh x + C}$$

(b) $\displaystyle\int \underset{u}{\sin x}\,\underset{dv}{\sec^2 x\,dx} = \underset{u}{\sin x}\,\underset{v}{\tan x} - \int \underset{v}{\tan x}\,\underset{du}{\cos x\,dx}$

$\tan x = \frac{\sin x}{\cos x}$

By parts

$u = \sin x \qquad v = \tan x$

$du = \cos x\,dx \qquad dv = \sec^2 x\,dx$

$$= \sin x \tan x - \int \sin x\,dx$$
$$= \boxed{\sin x \tan x + \cos x + C}$$

7. (10 pts) Find the area between the x-axis and the graph of the function $f(x) = 9x^2 \operatorname{Ln} x$, between $x = 1$ and $x = e$.

$3x^2$

$$\int_1^e 9x^2 \operatorname{Ln} x \, dx = \underset{v}{3x^3} \cdot \underset{u}{\operatorname{Ln} x}\Big]_1^e - \int_1^e \underbrace{\frac{3x^3}{x}}_{du} dx$$

By parts:

$u = \operatorname{Ln} x$ $\quad v = 3x^3$

$du = \frac{1}{x} dx$ $\quad dv = 9x^2 dx$

$$= 3 \cdot e^3 \operatorname{Ln} e - 3 \cdot 1^3 \cdot \operatorname{Ln} 1 - x^3\Big]_1^e$$

$$= 3e^3 - e^3 + 1$$

$$= \boxed{2e^3 + 1}$$

8. (10 pts) Use integration by parts to prove the reduction formula

$$\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx.$$

$u = x^n$ $\quad v = e^x$

$du = nx^{n-1}$ $\quad dv = e^x dx$

$$\boxed{\int \underset{u}{x^n} \underbrace{e^x dx}_{dv} = \underset{u}{x^n} \underset{v}{e^x} - \underbrace{n \int x^{n-1}}_{du} \overset{v}{e^x} dx}$$

DONE

Bonus:

1. (5 pts) What is wrong with the following argument?

   Theorem: $0 = 1$.

   Proof: Let us find $\int \frac{1}{x}\, dx$ as follows: Do integration by parts by taking

$$u = \frac{1}{x}, \quad dv = dx \quad \text{so that} \quad du = -\frac{1}{x^2}, \quad v = x$$

   to get

$$\int \frac{1}{x}\, dx = \frac{1}{x} \cdot x - \int -\frac{1}{x^2} \cdot x\, dx = 1 + \int \frac{1}{x}\, dx.$$

   This implies that

$$(*) \quad \int \frac{1}{x}\, dx = 1 + \int \frac{1}{x}\, dx,$$

   and therefore, subtracting $\int \frac{1}{x}\, dx$ from both sides, we get

$$0 = 1.$$

   "QED"

$\int f(x)\,dx$ means "an antiderivative of $f(x)$". But $f(x)$ has many antiderivatives (add any constant) so two functions can both be antiderivatives of f and yet be different (differ by a constant). Hence $(*)$ is not quite correct.

2. (5 pts) If f' is continuous, use L'Hôpital's rule to show that

$$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x).$$

$$\frac{d(f(x-h))}{dh} = f'(x-h)\cdot ?$$

$$\lim_{h\to 0} \frac{\overset{\to f(x)}{f(x+h)} - \overset{\to f(x)}{f(x-h)}}{\underset{\to 0}{2h}} \overset{LH}{=} \lim_{h\to 0} \frac{\overset{\to f'(x)}{f'(x+h)} + \overset{\to f'(x)}{f'(x-h)}}{2}$$

$$= \frac{2f'(x)}{2} = \boxed{f'(x)}$$