## Midterm 3. Calculus I - MATH31, Section D03. Spring 2025.

**Time allowed: 110 minutes** Professor: Luis Fernández

NAME:________________________________________

- The exam has SEVEN questions. Point values are indicated in each problem. Total is 106 points (6 are extra credit).
- Write your answers in the spaces provided. To get full credit you must show all your work.
- Please indicate your final answer clearly.
- No electronic devices besides a non-graphing calculator, or notes, are allowed.
- You will not be able to use the bathroom once the exam starts.

1. (*15 points*) Find the absolute maximum and minimum values of $f(x) = x + 2\cos(x)$ on the interval $[0, 2\pi]$.

Critical points: $f'(x) = 1 - 2\sin x = 0 \Rightarrow \sin x = \frac{1}{2}$

$\Rightarrow x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$.

Critical points: $\frac{\pi}{6}, \frac{5\pi}{6}$.

Endpoints: $0, 2\pi$.

Evaluate $f(x)$:

$$f(0) = 0 + 2\cos 0 = 2$$

$$f(2\pi) = 2\pi + 2\cos(2\pi) = 2\pi + 2 \approx 8.28$$

$$f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + 2\cos\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \frac{\cancel{2}\sqrt{3}}{\cancel{2}} = \frac{\pi}{6} + \sqrt{3} \approx 2.26$$

$$f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} + 2\cos\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} - \sqrt{3} \approx 0.88$$

Absolute max at $(2\pi, 2\pi + 2)$

Absolute min at $\left(\frac{5\pi}{6}, \frac{5\pi}{6} - \sqrt{3}\right)$.

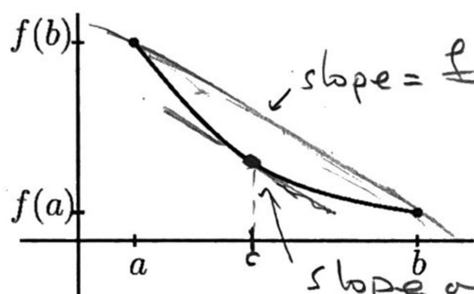
**2.** (*14 points*) Mean Value Theorem:

(a) State the Mean Value Theorem. Do not forget to write the hypothesis clearly.

Suppose $f$ is continuous in $[a,b]$ and differentiable in $(a,b)$. Then there is $c$ between $a$ and $b$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

(b) Use the graph of the function $f$ below to approximately find the point $c$ in the $x$-axis that satisfies the conclusion of the Mean Value Theorem.




slope $= \dfrac{f(b)-f(a)}{b-a}$

slope of tangent $= f'(c)$.

**3.** (*16 points*) Find an antiderivative of each of the following functions.

(a) $f(x) = \dfrac{2 - xe^x}{x} = \dfrac{2}{x} - \dfrac{\cancel{x}e^x}{\cancel{x}} = \dfrac{2}{x} - e^x.$

Antiderivative: $\boxed{2\ln x - e^x + C}$

(b) $f(x) = 2x^4 + 3x^3 - x^2$

Antiderivative: $\boxed{\dfrac{2x^5}{5} + \dfrac{3x^4}{4} - \dfrac{x^3}{3} + C}$

**4.** (*16 points*) Find the following limits. If it applies, you can use L'Hôpital's rule.

(a) $\lim\limits_{x\to 0} \dfrac{\sin 2x}{e^{3x}-1}$ $\overset{LH}{=} \lim\limits_{x\to 0} \dfrac{2\cos 2x}{3e^{3x}} = \boxed{\dfrac{2}{3}}$

(b) $\lim\limits_{x\to\infty} \left(1+\dfrac{5}{x}\right)^{\frac{x}{3}} = y$. Then $\ln y = \lim\limits_{x\to\infty} \dfrac{x}{3} \ln\left(1+\dfrac{5}{x}\right)$ $\to \infty \cdot 0$

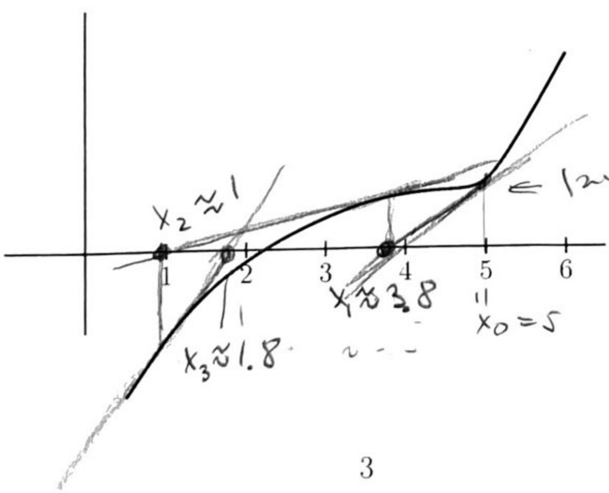
$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{5}{x}\right)}{\frac{3}{x}}$$

$$\overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{1+5/x} \cdot \frac{-5}{\cancel{x^2}}}{-\frac{3}{\cancel{x^2}}} = \frac{5}{3}.$$

Therefore, $\ln y = \dfrac{5}{3}$, so $y = e^{5/3}$, and

$$\lim_{x\to\infty} \left(1+\frac{5}{x}\right)^{\frac{x}{3}} = e^{5/3}$$

**5.** (*10 points*) The following picture is the graph of the function $f$. Illustrate the first 3 steps of Newton's method with initial point $x_0 = 5$ to find the solution of the equation $f(x) = 0$. Do a rough approximation of the values of $x_1$, $x_2$, $x_3$.



6. (*15 points*) Find the point $P$ on the graph of the function $y = \sqrt{x}$ that is closest to the point $(6, 0)$.

The point $P$ has the form $(x, \sqrt{x})$.

$$d(P, (6,0)) = \sqrt{(x-6)^2 + (\sqrt{x} - 0)^2} = \sqrt{(x-6)^2 + x}.$$

Note that minimizing $d$ is the same as minimizing $d^2$, so let us find the absolute minimum of $d^2 = (x-6)^2 + x$.

$$(d^2)' = 2(x-6) + 1 = 0 \Rightarrow 2x - 12 + 1 = 0 \Rightarrow x = \frac{11}{2} = \underline{\underline{5.5}}.$$

Note that

$(d^2)'(6) = 1$

$(d^2)'$

5    5.5    6

$(d')(5) = -2+1 = -1$    $-$    $+$

Therefore, [redacted] $(5.5, \sqrt{5.5})$ is the point where the absolute minimum is achieved, which is

$$\sqrt{(5.5-6)^2 + (\sqrt{5.5} - 0)^2} = \sqrt{0.25 + 5.5}$$

$$= \boxed{\sqrt{5.75}}.$$

**7.** (*24 points*) The function $f(x) = \dfrac{12x}{x^2+4}$ has the following properties:

- Domain: $(-\infty, \infty)$.
- Symmetry: it is an odd function.
- $f'(x) = \dfrac{-12(x^2-4)}{(x^2+4)^2}$ and $f''(x) = \dfrac{24x\,(x^2-12)}{(x^2+4)^3}$.

Find:

(a) The intercepts.

(b) The horizontal asymptotes.

(c) The intervals of increase/decrease.

(d) The local maximum and minimum values.

(e) The intervals where $f$ is concave up/concave down, and the points of inflection.

(f) Sketch the graph in the axes on the next page.

(a) $f(0) = \dfrac{12(0)}{(0)^2+4} = 0 \Rightarrow \boxed{y\text{-int is } (0,0)}$

$f(x) = \dfrac{12x}{x^2+4} = 0 \Rightarrow 12x = 0 \Rightarrow x = 0 \Rightarrow \boxed{x\text{-int is } (0,0)}$

(b) $\displaystyle\lim_{x\to\pm\infty} \frac{12x}{x^2+4} \overset{LH}{=} \lim_{x\to\pm\infty} \frac{12}{2x} = 0.$

H.A. at $\boxed{y=0}$

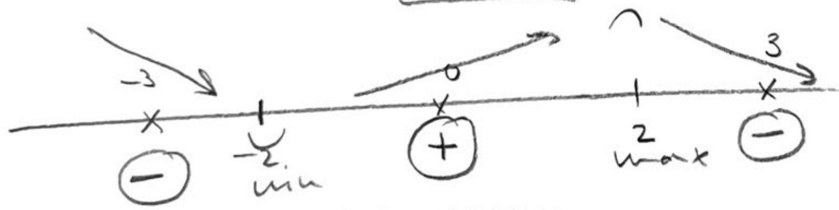
(c) $f'(x) = \dfrac{-12(x^2-4)}{(x^2+4)^2} = 0$

$\Rightarrow x^2-4=0 \Rightarrow \boxed{x = \pm 2}$

$f'(-3) = \dfrac{-12(9-4)}{\text{positive}} < 0$

$f'(0) = \dfrac{48}{\text{pos}} > 0$

$f'(3) = \dfrac{-12(9-4)}{\text{pos}} < 0$




$\boxed{\begin{array}{l}\text{Increasing: } (-2,2). \\ \text{Decreasing: } (-\infty,-2) \cup (2,\infty)\end{array}}$

(d) $\boxed{\text{Max at } \left(2, \tfrac{24}{8}\right) = (2,3)}$

$\boxed{\text{Min at } (-2,-3)}$

(e) $f''(x) = \dfrac{24x(x^2-12)}{(x^2+4)^3} = 0 \Rightarrow \underline{\underline{x=0}}$ or $x^2 = 12$

$\Rightarrow x = \pm\sqrt{12}$

$\Rightarrow x = \pm \underline{\underline{2\sqrt{3}}}$


$f''(-4) = \dfrac{24(-4)(16-12)}{\text{positive}} < 0$ $\quad$ $f''(-1) = \dfrac{24(-1)(1-12)}{\text{positive}} > 0$

$f''(4) = \dfrac{24(4)(16-12)}{\text{positive}} > 0$ $\quad$ $f''(1) = \dfrac{24(1)(1-12)}{\text{positive}} < 0$

Concave up: $(-2\sqrt{3}, 0) \cup (2\sqrt{3}, \infty)$

Concave down: $(-\infty, -2\sqrt{3}) \cup (0, 2\sqrt{3})$.

$f(2\sqrt{3}) = \dfrac{24\sqrt{3}}{12+4} = \dfrac{3\sqrt{3}}{2}$

Inflection points: $(0,0)$, $\left(2\sqrt{3}, \dfrac{3\sqrt{3}}{2}\right)$, $\left(-2\sqrt{3}, -\dfrac{3\sqrt{3}}{2}\right)$

$\approx (3.5, 2.6)$ $\quad$ $\approx (-3.5, -2.6)$