Midterm 2. Calculus I - MATH31, Section D03. Spring 2025.

Time allowed: 110 minutes Professor: Luis Fernández

NAME: SOLUTION

- The exam has SIX questions. Point values are indicated in each problem. Total is 100 points.
- Write your answers in the spaces provided. To get full credit you must show all your work.
- Please indicate your final answer clearly.
- No electronic devices besides a non-graphing calculator, or notes, are allowed.
- You will not be able to use the bathroom once the exam starts.

1. (*32 points*) Find the derivative of the following functions. Do not simplify the answer.

(a) $f(x) = e^4 + x^e$.

$\boxed{f'(x) = e x^{e-1}}$

(e) $f(x) = e^{\sqrt{x}} \sin(x)$.

$\boxed{f'(x) = \frac{e^{\sqrt{x}}}{2\sqrt{x}} \cdot \sin x + e^{\sqrt{x}} \cos x}$

(b) $f(x) = \dfrac{\sin(2x)}{x}$.

$\boxed{f'(x) = \frac{2\cos(2x)\cdot x - \sin(2x)}{x^2}}$

(f) $f(x) = x^{5x}$ (logarithmic differentiation).

Say $y = x^{5x}$, so $\ln y = 5x \ln x$.

$\frac{y'}{y} = 5\ln x + \frac{5x}{x} = 5\ln x + 5$

$\Rightarrow \boxed{(x^{5x})' = x^{5x} \cdot (5\ln x + 5)}$

(c) $f(x) = e^x \sqrt{x^2+10}$.

$\boxed{f'(x) = e^x \sqrt{x^2+10} + e^x \cdot \frac{2x}{2\sqrt{x^2+10}}}$

(g) $f(x) = \arctan(\ln x)$.

$\boxed{f'(x) = \frac{1}{(\ln x)^2+1} \cdot \frac{1}{x}}$.

(d) $f(x) = \ln(\sin(\cos(x^2)))$.

$\boxed{f'(x) = \frac{\cos(\cos(x^2)) \cdot (-\sin(x^2)) \cdot 2x}{\sin(\cos(x^2))}}$

(h) $f(x) = \tan(x)\sinh(2x)$

$\boxed{f'(x) = \sec^2(x)\sinh(2x) + \tan(x) \cdot 2\cosh(x)}$

2. (*15 points*) Use implicit differentiation to find an equation of the tangent line to the ellipse defined by $x^2 - 4xy + 5y^2 = 13$ at the point $(-1, -2)$.

$$\frac{d}{dx}(x^2 - 4xy + 5y^2) = \frac{d}{dx}(13)$$

$$2x - 4y - 4xy' + 10yy' = 0$$

$$x = -1,\ y = -2 \Rightarrow 2(-1) - 4(-2) - 4(-1)y' + 10(-2)y' = 0$$

$$\Rightarrow -2 + 8 + 4y' - 20y' = 0$$

$$6 - 16y' = 0 \Rightarrow y' = \frac{6}{16} = \boxed{\frac{3}{8}}$$

Equation

$$\boxed{y + 2 = \frac{3}{8}(x+1)}$$

3. (*15 points*) Use a linear approximation to find the approximate value of the number $\sqrt{23}$.

Use the function $f(x) = \sqrt{x}$ at $x = 25$.

$f'(x) = \frac{1}{2\sqrt{x}}$. $f(25) = \sqrt{25} = 5$.

$$f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{2 \cdot 5} = \frac{1}{10}.$$

$$L(x) = 5 + \frac{1}{10}(x - 25)$$

$$f(23) \approx L(23) = 5 + \frac{1}{10}(23 - 25) =$$

$$= 5 + \frac{1}{10}(-2)$$

$$= 5 - \frac{1}{5}$$

$$= \boxed{4.8}$$

4. (*20 points*) The length l of a rectangle is *increasing* at a rate of 4 cm/s. At the same time its height h is *decreasing* at a rate of 7 cm/s. At what rate is the area increasing (or decreasing) when the length is 10 cm and the height is 4 cm? [The area of a rectangle of length l and height h is $A = l \cdot h$.]

$$\frac{dl}{dt} = 4$$

$$\frac{dh}{dt} = -7$$

at $h = 4$

$l = 10$

$$A = l \cdot h$$

$$\Rightarrow \frac{dA}{dt} = \frac{dl}{dt} \cdot h + l \frac{dh}{dt}$$

$$\Rightarrow \frac{dA}{dt} = 4 \cdot 4 + 10 \cdot (-7)$$

$$= 16 - 70$$

$$= -55 \text{ cm/s}.$$

A decreases at a rate of 55 cm/s

5. (*8 points*) Find the following limits:

(a) $\lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)}$ $= \lim_{x\to 0} \frac{\sin(3x)}{3x} \cdot \frac{2x}{\sin(2x)} \cdot \frac{3}{2}$

$= \underbrace{\lim_{x\to 0} \frac{\sin(3x)}{3x}}_{= 1} \cdot \underbrace{\lim_{x\to 0} \frac{2x}{\sin(2x)}}_{= 1} \cdot \frac{3}{2} = 1 \cdot 1 \cdot \frac{3}{2}$

$= \boxed{\frac{3}{2}}$

(b) $\lim_{x\to\infty} \frac{\sinh(3x)}{e^{3x}}$ $= \lim_{x\to\infty} \frac{\frac{e^x - e^{-x}}{2}}{e^x} = \lim_{x\to\infty} \frac{\frac{e^x}{e^x} - \frac{e^{-x}}{e^x}}{2}$

$= \lim_{x\to\infty} \frac{1 - e^{-2x}}{2}$ $e^{-2x} \to 0$ $= \boxed{\frac{1}{2}}$

6. (*15 points*) Suppose that f and g are differentiable functions such that $f(g(x)) = x^3$. We only know that $g(3) = 7$ and that $f'(7) = -4$. Find $g'(3)$.

$(f(g(x)))' = (x^3)'$

$f'(g(x)) \cdot g'(x) = 3x^2$. $g(3) = 7$

At $x = 3$, $f'(g(3)) = f'(7) = \underline{\underline{-4}}$

$\Rightarrow (-4) \cdot g'(x) = 3(3)^2 = 27$

$\Rightarrow \boxed{g'(x) = -\frac{27}{4}}$