Midterm 1. Calculus I - MATH31, Section D03. V. 1. Spring 2025.

Time allowed: 110 minutes Professor: Luis Fernández

NAME: **SOLUTION**

The exam has FIVE questions. Point values are indicated in each problem, for a total of 100 points. Write your answers in the spaces provided. To get full credit you must show all your work. Please indicate your final answer clearly. No calculators, electronic devices, or notes are allowed.

1. (*18 points*) Use the graph of the function f given below to find the following. Note that arrows mean that the graph extends in that direction forever, and the asymptotes are represented by dashed lines.

(a) $\lim_{x\to -2^-} f(x) = 5$

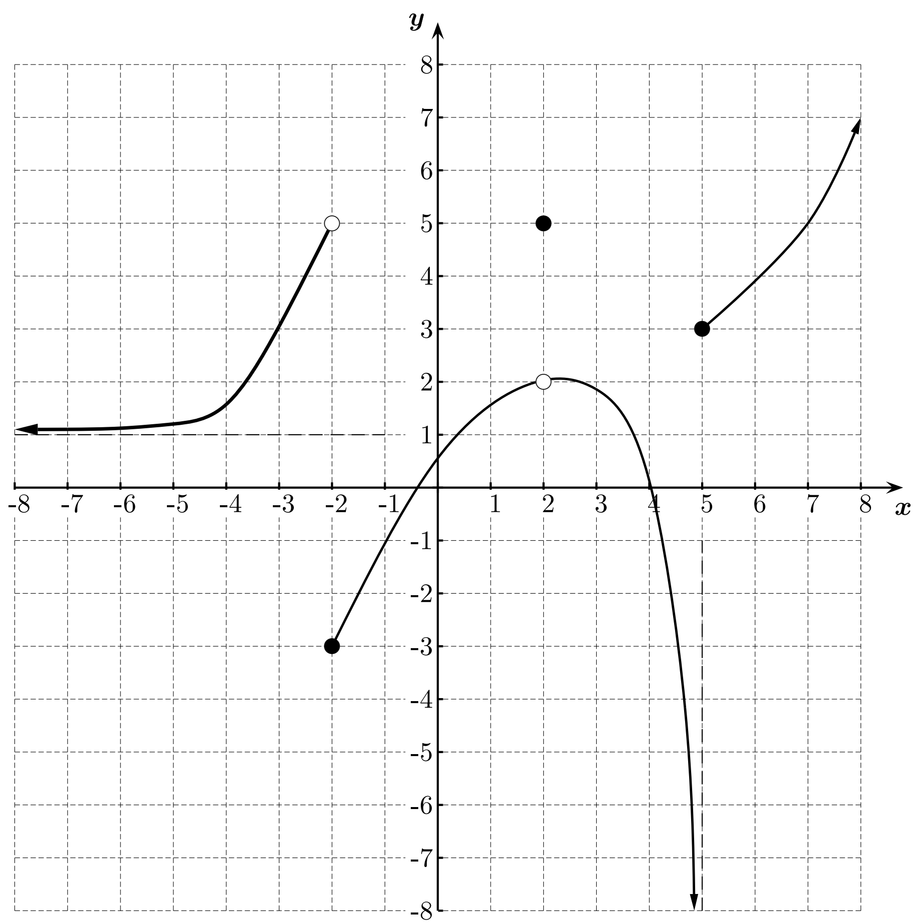
(b) $\lim_{x\to -2^+} f(x) = -3$

(c) $\lim_{x\to 5^-} f(x) = -\infty$

(d) $\lim_{x\to 5^+} f(x) = 3$

(e) $\lim_{x\to -\infty} f(x) = 1$

(f) At $x = -2$, is f continuous from the right, continuous from the left, or neither?
Answer: Continuous from the right because $\lim_{x\to -2^+} f(x) = -3 = f(-2)$. But not continuous from the left because $\lim_{x\to -2^-} f(x) = 5 \neq -3 = f(-2)$.



2. (*25 points*) Find the following limits, justifing your answer.

(a) $\lim_{x\to 4} \dfrac{x^2-3x-4}{x^2-7x+12} =$

Solution: If we evaluate we get $\frac{0}{0}$ which is an indeterminate form. Therefore we need to work a bit more. If we factor we get $\lim_{x\to 4} \dfrac{x^2-3x-4}{x^2-7x+12} = \lim_{x\to 4} \dfrac{(x-4)(x+1)}{(x-4)(x-3)} = \lim_{x\to 4} \dfrac{(x+1)}{(x-3)} = \dfrac{5}{1} = 5.$

(b) $\lim_{x\to\infty} \dfrac{3x^2+2x-5}{2x^2-x+2} =$

Solution: If we evaluate we get $\frac{\infty}{\infty}$, which is an indeterminate form. So we do something more. Divide numerator and denominator by x^2 to get $\lim_{x\to\infty} \dfrac{3+\frac{2}{x}-\frac{5}{x^2}}{2-\frac{1}{x}+\frac{2}{x^2}} = \dfrac{3+0-0}{2-0+0} = \dfrac{3}{2}.$

(c) $\lim_{x\to 2^-} \dfrac{2x}{x^2-4} =$

Solution: $\lim_{x\to 2^-} \dfrac{2x}{x^2-4} = -\infty$, because when $x<2$, the numerator is positive and the denominator is negative, and when x approaches 2 from the left, the numerator approaches 4 and the denominator approaches 0.

(d) $\lim\limits_{x \to 0} x^2 \sin\left(\dfrac{1}{x^2}\right) =$

Solution: Since $-1 \leq \sin\left(\frac{1}{x^2}\right) \leq 1$, we have that $-x^2 \leq x^2 \sin\left(\frac{1}{x^2}\right) \leq x^2$. Since $\lim_{x\to 0}(-x^2) = 0$ and $\lim_{x\to 0}(x^2) = 0$, the Squeeze Theorem implies that $\lim\limits_{x \to 0} x^2 \sin\left(\dfrac{1}{x^2}\right) = 0$.

(e) $\lim\limits_{x \to 1} \dfrac{\sqrt{x+8} - 3}{x^2 - 1} =$

Solution: If you evaluate you get $\frac{0}{0}$, which is an indeterminate form, so we have to work a bit more. Since there is a radical, let us multiply by the conjugate of the numerator to get rid of it.

$$\lim_{x \to 1} \frac{\sqrt{x+8} - 3}{x^2 - 1} = \lim_{x \to 1} \frac{(\sqrt{x+8} - 3)(\sqrt{x+8} + 3)}{(x^2 - 1)(\sqrt{x+8} + 3)} = \lim_{x \to 1} \frac{(x+8) - 9}{(x-1)(x+1)(\sqrt{x+8} + 3)} = \lim_{x \to 1} \frac{(x-1)}{(x-1)(x+1)(\sqrt{x+8}}$$

$$\lim_{x \to 1} \frac{1}{(x+1)(\sqrt{x+8} + 3)} = \frac{1}{2 \cdot (\sqrt{9} + 3)} = \frac{1}{12}.$$

3. (*17 points*) The figure below represents the graph of a function f, together with the tangent lines to the graph at the points with $x = -6, -4, -2, 1,$ and 4.

(a) (*2 points*) Find $f'(-6)$. Answer: -1

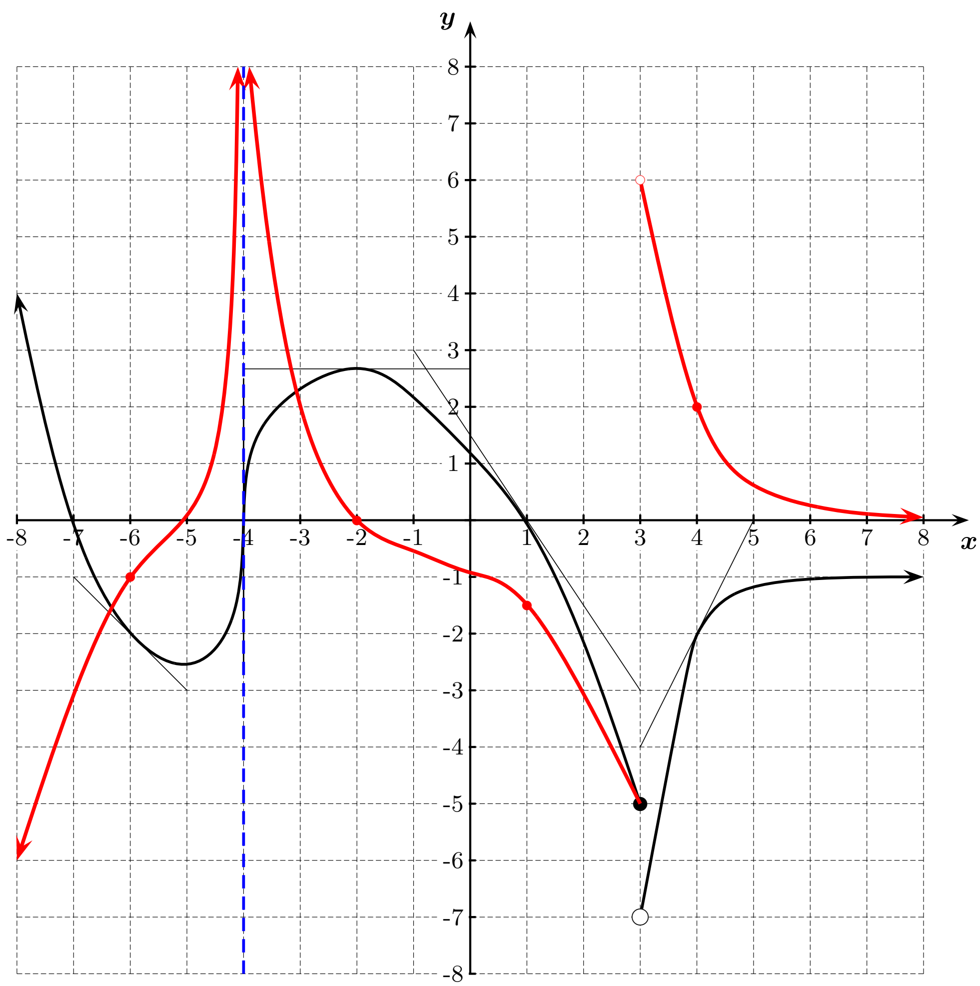
(b) (*2 points*) Find $f'(-2)$. Answer: 0

(c) (*2 points*) Find $f'(1)$. Answer: $-\frac{3}{2}$

(d) (*2 points*) Find $f'(4)$. Answer: 2

(e) (*2 points*) Find the values of x where $f'(x)$ does not exist.
Answer: At $x = -4$ and $x = 3$

(f) (*7 points*) Sketch the graph of $f'(x)$ in the same coordinate system.



4. (*20 points*) Let $f(x) = \dfrac{x+5}{x+1}$.

(a) (*5 points*) Write the definition of the derivative of a function f at the point b:

$$f'(b) = \lim_{x\to b} \frac{f(x)-f(b)}{x-b} \quad \text{or} \quad \lim_{h\to 0} \frac{f(b+h)-f(b)}{h}$$

(b) (*10 points*) Use the definition of derivative to find $f'(1)$.

Solution: $\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{x\to 0} \dfrac{\frac{(1+h)+5}{(1+h)+1} - \frac{1+5}{1+1}}{h} = \lim_{x\to 0} \dfrac{\frac{h+6}{h+2} - \frac{3}{1}}{h} = \lim_{x\to 0} \dfrac{\frac{h+6}{h+2} - \frac{3(h+2)}{(h+2)}}{h} = \lim_{x\to 0} \dfrac{\frac{-2h}{h+2}}{h} =$ $\lim_{x\to 0} \dfrac{-2h}{h(h+2)} = \lim_{x\to 0} \dfrac{-2}{(h+2)} = \dfrac{-2}{2} = -1.$

(c) (*5 points*) Use part (b) to find the equation of the tangent line to the graph of f at the point (1,3).

Solution: The equation of the tangent line is $y - 3 = -1(x-1)$, or $y = -x + 4$.

5. (*20 points*) Given the function $$g(x) = \begin{cases} ax^2 - 4 & \text{if } x < 2 \\ x + a & \text{if } x \geq 2 \end{cases},$$

find the value of a so that g is continuous at $x = 2$.

Solution: Let us compute for which values of a is $\lim_{x \to 2} g(x) = g(2)$. Since the definition of the function changes at 2, let us do one sided limits.

$\lim\limits_{x \to 2^-} g(x) = \lim\limits_{x \to 2^-} (ax^2 - 4) = 4a - 4.$

$\lim\limits_{x \to 2^+} g(x) = \lim\limits_{x \to 2^+} (x + a) = a + 2.$

$g(2) = a + 2$ (the function is continuous from the right already, no matter what a is).

For g to be continuous at 2 we need $\lim_{x \to 2^-} g(x) = \lim_{x \to 2^+} g(x) = g(2)$. Therefore we need $4a - 4 = a + 2$, which gives $3a = 6$, or $a = 2$.

Thus, the function is continuous when $a = 2$.

6. BONUS EXERCISE. (*5 points*) Attempt ONLY when you have finished all the other exercises.

Let f be a continuous odd function. What is the value of $f(0)$? **Prove** carefully that the value you wrote is correct.