Review for Midterm 2
Solutions

Some exercises were written by Nikos Apostolakis and Quanlei Fang.)

This is a list of exercises to review the topics that will be in the midterm. If you know how to answer these questions then you will do very well in the exam.

1. Differentiate the following functions:

(a) $f(x) = \cot^2(x) + \pi^2$ (b) $f(x) = \sin(\tan(2x))$ (c) $f(x) = \dfrac{x^2-1}{e^x}$

(d) $f(x) = x(\ln x)^3$ (e) $f(x) = \cos(\ln x)$ (f) $f(x) = \log_2\left(\left(x^2+1\right)^5 \sin(5x)\right)$

(g) $f(x) = \sinh(2x+5)$ (h) $f(x) = 2^{\sqrt{x}}\sin(x)$ (i) $f(x) = \dfrac{1}{\arcsin(2x)}$

(j) $f(x) = e^{2x}\cosh x$ (k) $f(x) = \sqrt[6]{\dfrac{x+1}{x-1}}$ (l) $f(x) = x\arctan(5x)$

(m) $f(x) = x^{e^x}$ (n) $f(x) = x^{\sin x}$

Solutions:

(a) $f(x) = \cot^2(x) + \pi^2 \;\Rightarrow\; f'(x) = -2\cot(x)\csc^2(x)$

(b) $f(x) = \sin(\tan(2x)) \;\Rightarrow\; f'(x) = \cos(\tan(2x)) \cdot 2\sec^2(2x)$

(c) $f(x) = \frac{x^2-1}{e^x} \;\Rightarrow\; f'(x) = \frac{1-x^2+2x}{e^x}$

(d) $f(x) = x(\ln(x))^3 \;\Rightarrow\; f'(x) = (\ln(x))^3 + 3(\ln(x))^2$

(e) $f(x) = \cos(\ln(x)) \;\Rightarrow\; f'(x) = -\sin(\ln(x)) \cdot \frac{1}{x}$

(f) $f(x) = \log_2\left((x^2+1)^5\sin(5x)\right) \;\Rightarrow\; f'(x) = \frac{1}{\ln 2}\left(\frac{10x}{x^2+1} + \frac{5\cos(5x)}{\sin(5x)}\right)$

(g) $f(x) = \sinh(2x+5) \;\Rightarrow\; f'(x) = 2\cosh(2x+5)$

(h) $f(x) = 2^{\sqrt{x}}\sin(x) \;\Rightarrow\; f'(x) = 2^{\sqrt{x}}\left(\ln 2 \cdot \frac{1}{2\sqrt{x}}\sin(x) + \cos(x)\right)$

(i) $f(x) = \frac{1}{\arcsin(2x)} \;\Rightarrow\; f'(x) = -\frac{2}{\arcsin(2x)^2\sqrt{1-4x^2}}$

(j) $f(x) = e^{2x}\cosh(x) \;\Rightarrow\; f'(x) = e^{2x}(2\cosh(x) + \sinh(x))$

(k) $f(x) = \sqrt[6]{\frac{x+1}{x-1}} \;\Rightarrow\; f'(x) = \frac{1}{6}\left(\frac{x+1}{x-1}\right)^{-\frac{5}{6}} \cdot \frac{-2}{(x-1)^2}$

(l) $f(x) = x\arctan(5x) \;\Rightarrow\; f'(x) = \arctan(5x) + x \cdot \frac{5}{1+25x^2}$

(m) $f(x) = x^{e^x} \;\Rightarrow\; f'(x) = x^{e^x}\left(e^x\ln x + \frac{e^x}{x}\right)$

(n) $f(x) = x^{\sin x} \;\Rightarrow\; f'(x) = x^{\sin x}\left(\cos x \cdot \ln x + \frac{\sin x}{x}\right)$

2. Find the following limits:

(a) $\lim_{x\to 0} \dfrac{\sin(2x)}{\sin(3x)}$ (b) $\lim_{x\to\infty} \dfrac{\sinh(2x)}{\cosh(3x)}$

Solutions:

(a) $\lim_{x\to 0} \dfrac{\sin(2x)}{\sin(3x)} = \lim_{x\to 0} \dfrac{\sin(2x)}{2x} \cdot \dfrac{3x}{\sin(3x)} \cdot \dfrac{2}{3} = \dfrac{2}{3}$

Here we have used that $\lim_{t\to 0} \dfrac{\sin(t)}{t} = 1$ and $\lim_{t\to 0} \dfrac{t}{\sin(t)} = 1$.

(In this case, $t = 2x$ in the first limit and $t = 3x$ in the second limit.)

(b) $\lim_{x\to\infty} \frac{\sinh(2x)}{\cosh(3x)}$

Recall the definitions of the hyperbolic sine and cosine functions:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

Therefore

$$\lim_{x\to\infty} \frac{\sinh(2x)}{\cosh(3x)} = \lim_{x\to\infty} \frac{\frac{e^{2x}-e^{-2x}}{2}}{\frac{e^{3x}+e^{-3x}}{2}} = \lim_{x\to\infty} \frac{e^{2x} - e^{-2x}}{e^{3x} + e^{-3x}}.$$

When $x \to \infty$, $e^{2x} \to \infty$, $e^{3x} \to \infty$, $e^{-2x} \to 0$ and $e^{-3x} \to 0$. Therefore we would get $\frac{\infty}{\infty}$. To "tame the infinities", divide numerator and denominator by e^{3x} to get

$$\lim_{x\to\infty} \frac{\sinh(2x)}{\cosh(3x)} = \lim_{x\to\infty} \frac{\frac{e^{2x}-e^{-2x}}{e^{3x}}}{\frac{e^{3x}+e^{-3x}}{e^{3x}}} = \lim_{x\to\infty} \frac{\frac{e^{2x}}{e^{3x}} - \frac{e^{-2x}}{e^{3x}}}{\frac{e^{3x}}{e^{3x}} + \frac{e^{-3x}}{e^{3x}}} = \lim_{x\to\infty} \frac{e^{-x} - e^{-5x}}{1 + e^{-6x}} = \frac{0+0}{1+0} = 0.$$

3. Find an equation of the tangent to the curve at the given point:

   (a) $y = 4\sin^2 x$, at the point $\left(\frac{\pi}{6}, 1\right)$.

   Solution. We have
   $$\frac{dy}{dx} = 8\sin x \cos x,$$
   and at $x = \frac{\pi}{6}$ we have
   $$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{6}} = 8\sin\frac{\pi}{6}\cos\frac{\pi}{6} = 8\frac{1}{2}\frac{\sqrt{2}}{2} = 4\sqrt{2}.$$
   So the equation of the tangent line is
   $$y - 1 = 4\sqrt{2}\left(x - \frac{\pi}{6}\right).$$
   $\square$

(b) $y = \dfrac{x^2 - 4}{x^2 + 4}$, at the point $(0, -1)$.

Solution. We have

$$\frac{dy}{dx} = \frac{2x(x^2+4) - 2x(x^2-4)}{(x^2+4)^2} = \frac{-8x}{(x^2+4)^2},$$

$\square$

and at $x = 0$ we get

$$\left.\frac{dy}{dx}\right|_{x=0} = 0.$$

Thus we have a horizontal tangent line

$$y = -1.$$

(c) $y = \sqrt{4 - 2\sin x}$, at the point $(0, 2)$.

Solution. We have

$$\frac{dy}{dx} = \frac{-2\cos x}{2\sqrt{4 - 2\sin x}} = -\frac{\cos x}{\sqrt{4 - 2\sin x}},$$

and at $x = 0$ we get

$$\left.\frac{dy}{dx}\right|_{x=0} = -\frac{1}{2}.$$

So the tangent line is

$$y = -\frac{x}{2} + 2.$$

$\square$

(d) $x^3 + 3x^2y - 2xy^2 - y^3 = 49$, at the point $(3, 2)$.

Solution. We use implicit differentiation

$$3x^2 + 6xy + 3x^2y' - 2y^2 - 4xyy' - 3y^2y' = 0$$

and after substituting $x = 3$, $y = 2$ we get

$$27 + 36 + 27y' - 8 - 24y' - 12y' = 0$$

or equivalently

$$y' = \frac{55}{9}.$$

The tangent line then is

$$y - 2 = \frac{55}{9}(x - 3).$$

$\square$

(e) $x^{2/3} + y^{2/3} = 4$, at the point $(-3\sqrt{3}, 1)$.

Solution. Using implicit differentiation we have

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0,$$

and solving for y' gives

$$y' = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}}.$$

At the point $(-3\sqrt{3}, 1)$ then we get

$$\left.\frac{dy}{dx}\right|_{\substack{x=-3\sqrt{3}\\ y=1}} = -\frac{1}{\sqrt[3]{-3\sqrt{3}}} = \frac{1}{\sqrt[6]{27}}.$$

So the tangent line is

$$y - 1 = \frac{x + 3\sqrt{3}}{\sqrt[6]{27}}.$$

□

(f) $y \sin x^2 = x \sin y^2$, at the point $(\sqrt{\pi}, \sqrt{\pi})$.

Solution. Using implicit differentiation we get

$$y' \sin x^2 + 2xy \cos x^2 = \sin y^2 + \left(2xy \cos y^2\right) y',$$

and so at the point $(\sqrt{\pi}, \sqrt{\pi})$ we get

$$y' \sin \pi + 2\pi \cos \pi = \sin \pi + (2\pi \cos \pi)\, y'$$

or equivalently,

$$-2\pi = -2\pi y'.$$

So

$$\left.\frac{dy}{dx}\right|_{\substack{x=\sqrt{\pi}\\ y=\sqrt{\pi}}} = 1.$$

Thus the tangent line is

$$y - \sqrt{\pi} = x - \sqrt{\pi}$$

or equivalently

$$y = x.$$

□

(g) $y = e^{2x-1}$ at the point $(\frac{1}{2}, 1)$.

Solution. We have

$$\frac{dy}{dx} = 2e^{2x-1}$$

and so at $x = \frac{1}{2}$ we get

$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = 2.$$

The tangent line is therefore

$$y - 1 = 2\left(x - \frac{1}{2}\right)$$

or equivalently

$$y = 2x.$$

$\square$

(h) $y = \ln(2x^2 - x)$ at the point $(1, 0)$.

Solution. We have

$$\frac{dy}{dx} = \frac{4x-1}{2x^2-x},$$

and so at $x = 1$ we get

$$\left.\frac{dy}{dx}\right|_{x=1} = 3.$$

So the tangent line is

$$y = 3x - 3.$$

$\square$

4. At what points on the curve $y = \sin x - \cos x$, $0 \leq x \leq 2\pi$ is the tangent line horizontal?

Solution. Let $f(x) = \sin x - \cos x$. Then the tangent line is horizontal when $f'(x) = 0$. We have

$$f'(x) = \cos x + \sin x.$$

So

$$\begin{aligned} f'(x) = 0 &\iff \cos x + \sin x = 0 \\ &\iff \cos x = -\sin x \\ &\iff \tan x = -1 \\ &\iff x = \frac{3\pi}{4}, \text{ or } x = \frac{7\pi}{4}. \end{aligned}$$

So the tangent line is horizontal at the points $(\frac{3\pi}{4}, \sqrt{2})$ and $(\frac{7\pi}{4}, -\sqrt{2})$. $\square$

5. Find the points on the ellipse $2x^2 + y^2 = 1$ where the tangent line has slope 1.

Solution. Using implicit differentiation we get

$$4x + 2yy' = 0$$

which gives

$$y' = -\frac{2x}{y}.$$

Thus the tangent line has slope 1 at those points (x, y) of the ellipse where

$$-\frac{2x}{y} = 1$$

or equivalently, where

$$y = -2x.$$

Since these points lie in the ellipse $2x^2 + y^2 = 1$ we get

$$2x^2 + (-2x)^2 = 1 \iff 6x^2 = 1 \iff x = \pm\frac{1}{\sqrt{6}}.$$

So the tangent line has slope 1 at the points $(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}})$ and $(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})$. $\square$

6. Given a particle moving according to $s(t) = t^3 - 12t^2 + 36$, $t \geq 0$.

   (a) Find the velocity function and acceleration function.

   (b) What is the velocity after 3 seconds?

   (c) When is the particle at rest?

   (d) When is the particle moving forward? When is the particle moving backward?

   (e) Find the total distance the particle traveled in the first 8 seconds.

Solution:

Given that the particle's position function is $s(t) = t^3 - 12t^2 + 36$ for $t \geq 0$,

(a) 1. **Velocity** is the derivative of the position function $s(t)$:

$$v(t) = \frac{ds(t)}{dt} = \frac{d}{dt}(t^3 - 12t^2 + 36) = 3t^2 - 24t$$

So, the velocity function is: $v(t) = 3t^2 - 24t$

2. **Acceleration** is the derivative of the velocity function $v(t)$:

$$a(t) = \frac{dv(t)}{dt} = \frac{d}{dt}(3t^2 - 24t) = 6t - 24$$

Thus, the acceleration function is: $a(t) = 6t - 24$

(b) What is the velocity after 3 seconds?

To find the velocity at $t = 3$, substitute $t = 3$ into the velocity function:

$$v(3) = 3(3)^2 - 24(3) = 3(9) - 72 = 27 - 72 = -45$$

So, the velocity after 3 seconds is -45 units per second.

(c) When is the particle at rest?

A particle is at rest when its velocity is 0. Set $v(t) = 0$:

$$3t^2 - 24t = 0$$

Factor the equation:

$$3t(t - 8) = 0$$

So, $t = 0$ or $t = 8$.

Thus, the particle is at rest at $t = 0$ and $t = 8$.

(d) When is the particle moving forward? When is the particle moving backward?

- A particle is moving forward when its velocity is positive ($v(t) > 0$). - A particle is moving backward when its velocity is negative ($v(t) < 0$).

We already have the velocity function:

$$v(t) = 3t^2 - 24t$$

Factor it:

$$v(t) = 3t(t - 8)$$

To determine when $v(t) > 0$ or $v(t) < 0$, note that $v(t)$ is a parabola that opens up and has x-intercepts at 0 and 8 (do a picture). We see that $v(t) < 0$ between $t = 0$ and $t = 8$ and $v(t) > 0$ when $t > 8$.

Thus, the particle is moving: - **Backward** when $0 < t < 8$ - **Forward** when $t > 8$

(e) Find the total distance the particle traveled in the first 8 seconds.

To find the total distance traveled, we need to consider any changes in direction. From part (c), we know the particle is at rest at $t = 0$ and $t = 8$, and the particle changes direction at $t = 8$.

We will calculate the distance traveled from $t = 0$ to $t = 8$, taking into account any changes in direction.

1. **Position at $t = 0$:**

$$s(0) = (0)^3 - 12(0)^2 + 36 = 36$$

2. **Position at $t = 8$:**

$$s(8) = (8)^3 - 12(8)^2 + 36 = 512 - 768 + 36 = -220$$

Thus, the distance traveled between $t = 0$ and $t = 8$ is:

$$|s(8) - s(0)| = |-220 - 36| = |-256| = 256$$

Thus, the total distance traveled by the particle in the first 8 seconds is $\boxed{256}$ units.

7. A ladder 5 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 3 m/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 4 m from the wall?

Solution. Let x and y be the distance of the bottom and top, respectively, of the ladder from the wall, see Figure 1. Then because the wall is vertical we can apply the Pythagorean Theorem to get that

$$x^2 + y^2 = 25. \tag{1}$$

Differentiating Equation (1) with respect to time t we get

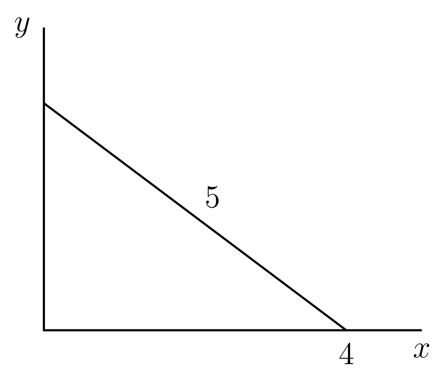

Figure 1: Question 7.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Longleftrightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \tag{2}$$

Now when $x = 4$ Equation (1) gives that $y = 3$, and since we know that $\frac{dx}{dt} = 3$ we can substitute in Equation (2) to get

$$6 + 4\frac{dy}{dt} = 0 \Longleftrightarrow \frac{dy}{dt} = -\frac{3}{2}.$$

Thus when the bottom of the ladder is 3 m away from the wall the bottom slides down at a rate of 1.5 m/s. □

8. Boat A travels west at a 50 miles per hour and boat B travels north at 60 miles per hour. The two boats are going to collide in 3 hours. At what rate are the two boats approaching each other 1 hour before the collision?

Solution. Let x, and y, respectively, be the distance of boat A and B from the point of collision O, see Figure 2. Then the distance z of the two boats satisfies the equation

$$z^2 = x^2 + y^2. \tag{3}$$

Differentiating Equation (3) with respect to time t gives

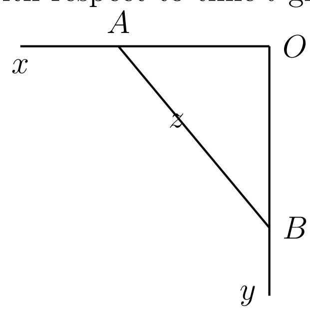

Figure 2: Question 8.

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \iff z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} \tag{4}$$

One hour before the collision boat A is 50 miles away from O, and boat B is 60 miles away. By Equation (3) their distance is then

$$z = \sqrt{50^2 + 60^2} = 10\sqrt{61}.$$

We also know that $\frac{dx}{dt} = -50\,\mathrm{mi/h}$ and $\frac{dy}{dt} = -60\,\mathrm{mi/h}$ so substituting in Equation (4) we get

$$10\sqrt{61}\frac{dz}{dt} = 50(-50) - 60(-60) \iff \frac{dz}{dt} = -10\sqrt{61}.$$

Therefore one hour before the collision the two boats are approaching each other at a rate of $10\sqrt{61} \approx 78.1\,\mathrm{mi/h}$. □

9. The surface area of a cube is increasing at a rate of $2\,\mathrm{cm}^2/\mathrm{min}$. How fast is the volume of the cube increasing when the length of the edge is $20\,\mathrm{cm}$?

Solution. If x is the length of an edge of the cube then its volume is

$$V(x) = x^3,$$

and each of its six faces has area x^2. The surface area of the cube is therefore given by

$$A(x) = 6x^2.$$

Differentiating these equations with respect to time t gives

$$\frac{dV(x)}{dt} = 3x^2\frac{dx}{dt}, \qquad \frac{dA(x)}{dt} = 12x\frac{dx}{dt}.$$

Now we know that $\frac{dA(x)}{dt} = 2\,\text{cm}^2/\text{min}$ and so by substituting in the second equation above, when $x = 20\,\text{cm}$ we have that

$$2 = 240\frac{dx}{dt} \iff \frac{dx}{dt} = \frac{1}{120}.$$

Substituting in the first equation then gives

$$\frac{dV(x)}{dt} = 3(20)^2\frac{1}{120} = 10.$$

Thus when the length of the edge is $20\,\text{cm}$ the volume of the cube increases at a rate of $10\,\text{cm}^3/\text{min}$. □

10. Use appropriate linear approximations to estimate the following:

A. $\sqrt{9.04}$ B. $\sin 0.02$ C. $(1.03)^{-1/3}$ D. $\sqrt[3]{0.97}$

Solution. (a) Let $f(x) = \sqrt{x}$, and $L(x)$ its linear approximation at $a = 9$. We have $f(9) = 3$ and $f'(x) = \frac{1}{2\sqrt{x}}$. So

$$f'(9) = \frac{1}{6}.$$

Therefore

$$L(x) = 3 + \frac{x-9}{6}.$$

Now

$$\sqrt{9.04} = f(9.04) \approx L(9.04) = 3 + \frac{0.04}{6} \approx 3.0067.$$

(b) Let $f(x) = \sin x$, and $L(x)$ its linear approximation at $a = 0$. We have $f(0) = 0$ and $f'(x) = \cos x$. So

$$f'(0) = 1.$$

Therefore

$$L(x) = x.$$

Now

$$\sin 0.02 = f(0.02) \approx L(0.02) = 0.02.$$

(c) Let $f(x) = x^{-1/3}$, and $L(x)$ its linear approximation at $a = 1$. We have $f(1) = 1$ and $f'(x) = -\frac{1}{3}x^{-4/3}$. So

$$f'(1) = -\frac{1}{3}.$$

Therefore

$$L(x) = 1 - \frac{x-1}{3}.$$

Now

$$(1.03)^{-1/3} = f(1.03) \approx L(1.03) = 1 - \frac{0.03}{3} \approx 0.99.$$

(d) Let $f(x) = \sqrt[3]{x}$, and $L(x)$ its linear approximation at $a = 1$. We have $f(1) = 1$ and $f'(x) = \frac{1}{3}x^{-2/3}$. So

$$f'(1) = \frac{1}{3}.$$

Therefore

$$L(x) = 1 + \frac{x-1}{3}.$$

Now

$$\sqrt[3]{0.97} = f(0.97) \approx L(0.97) = 1 + \frac{-0.03}{3} \approx 0.99.$$

$\square$