Midterm 2. Calculus I - MATH31, Section D03. Practice test. Spring 2025.

Time allowed: 110 minutes Professor: Luis Fernández

NAME: **SOLUTION**

- The exam has FIVE questions. Point values are indicated in each problem, for a total of 100 points.
- Write your answers in the spaces provided. To get full credit you must show all your work.
- Please indicate your final answer clearly.
- No electronic devices besides a non-graphing calculator, or notes, are allowed.
- You will not be able to use the bathroom once the exam starts.

1. (*32 points*) Find the derivative of the following functions. You do not need to simplify the answer.

(a) $f(x) = \pi^4 + x^\pi$.

(b) $f(x) = \dfrac{\sin(2x)}{1+\cos(2x)}$.

(c) $f(x) = (x^2+10)e^{2x}$.

(d) $f(x) = \ln\left(x + \sqrt{x^2-1}\right)$.

(e) $e^{\sqrt{x}}\cos(x)$.

(f) $f(x) = x^x$ (logarithmic differentiation).

(g) $f(x) = \dfrac{1}{\arctan(2x+5)}$

(h) $f(x) = e^{2x}\sinh 2x$

Solution:

(a) $f(x) = \pi^4 + x^\pi$

The derivative of the constant π^4 is zero. For x^π, we apply the power rule:

$$f'(x) = \pi x^{\pi-1}$$

(b) $f(x) = \frac{\sin(2x)}{1+\cos(2x)}$

We use the quotient rule for derivatives: $\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{v(x)u'(x)-u(x)v'(x)}{v(x)^2}$. Let $u(x) = \sin(2x)$ and $v(x) = 1+\cos(2x)$, and compute their derivatives:

$$u'(x) = 2\cos(2x), \quad v'(x) = -2\sin(2x)$$

So,

$$f'(x) = \frac{(1+\cos(2x))\cdot 2\cos(2x) - \sin(2x)\cdot(-2\sin(2x))}{(1+\cos(2x))^2}$$

(c) $f(x) = (x^2+10)e^{2x}$

We apply the product rule: $\frac{d}{dx}\left(u(x)v(x)\right) = u'(x)v(x) + u(x)v'(x)$. Let $u(x) = x^2+10$ and $v(x) = e^{2x}$, so:

$$u'(x) = 2x, \quad v'(x) = 2e^{2x}$$

Therefore,

$$f'(x) = 2xe^{2x} + (x^2+10)\cdot 2e^{2x}$$

(d) $f(x) = \ln\left(x+\sqrt{x^2-1}\right)$

We apply the chain rule. Let $u(x) = x+\sqrt{x^2-1}$, then:

$$f'(x) = \frac{1}{x+\sqrt{x^2-1}}\cdot\left(1+\frac{d}{dx}\left(\sqrt{x^2-1}\right)\right)$$

The derivative of $\sqrt{x^2-1}$ is $\frac{x}{\sqrt{x^2-1}}$. Thus:

$$f'(x) = \frac{1}{x+\sqrt{x^2-1}}\cdot\left(1+\frac{x}{\sqrt{x^2-1}}\right)$$

(e) $f(x) = e^{\sqrt{x}} \cos(x)$
We use the product rule. Let $u(x) = e^{\sqrt{x}}$ and $v(x) = \cos(x)$, so:

$$u'(x) = \frac{1}{2\sqrt{x}} e^{\sqrt{x}}, \quad v'(x) = -\sin(x)$$

Therefore:

$$f'(x) = \frac{1}{2\sqrt{x}} e^{\sqrt{x}} \cos(x) - e^{\sqrt{x}} \sin(x)$$

(f) $f(x) = x^x$ (Logarithmic differentiation)
To differentiate $f(x) = x^x$, we take the natural logarithm of both sides:

$$\ln(f(x)) = \ln(x^x) = x \ln(x)$$

Differentiate both sides with respect to x:

$$\frac{1}{f(x)} f'(x) = \ln(x) + 1$$

Therefore:

$$f'(x) = x^x \left(\ln(x) + 1\right)$$

(g) $f(x) = \frac{1}{\arctan(2x+5)}$
We apply the chain rule. Let $u(x) = \arctan(2x + 5)$, so:

$$f'(x) = -\frac{1}{u(x)^2} \cdot \frac{d}{dx} \left(\arctan(2x + 5)\right)$$

The derivative of $\arctan(2x + 5)$ is $\frac{2}{(2x+5)^2+1}$. Therefore:

$$f'(x) = -\frac{1}{\left(\arctan(2x + 5)\right)^2} \cdot \frac{2}{(2x + 5)^2 + 1}$$

(h) $f(x) = e^{2x} \sinh(2x)$
We apply the product rule. Let $u(x) = e^{2x}$ and $v(x) = \sinh(2x)$, so:

$$u'(x) = 2e^{2x}, \quad v'(x) = 2\cosh(2x)$$

Therefore:

$$f'(x) = 2e^{2x} \sinh(2x) + e^{2x} \cdot 2\cosh(2x)$$

Simplifying:

$$f'(x) = 2e^{2x} \left(\sinh(2x) + \cosh(2x)\right)$$

2. (*15 points*) Use implicit differentiation to find an equation of the tangent line to the ellipse defined by $3x^2 + 4xy + 5y^2 = 37$ at the point $(4, -1)$.

Solution:

Differentiate both sides of the equation implicitly:

$$\frac{d(3x^2 + 4xy + 5y^2)}{dx} = \frac{d(37)}{dx} \Rightarrow \frac{d(3x^2)}{dx} + 4\frac{d(xy)}{dx} + 5\frac{d(y^2)}{dx} = \frac{d(37)}{dx} \Rightarrow 6x + \frac{d(4x)}{dx} \cdot y + 4x \cdot \frac{dy}{dx} + 10y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 6x + 4y + 4x \cdot \frac{dy}{dx} + 10y\frac{dy}{dx} = 0.$$

Then evaluate at the point $x = 4$, $y = -1$ to get $6(4) + 4(-1) + 4(4) \cdot \frac{dy}{dx} + 10(-1)\frac{dy}{dx} = 0 \Rightarrow 20 + 6 \cdot \frac{dy}{dx} = 0.$

Solve for $\frac{dy}{dx}$ to get $\frac{dy}{dx} = -\frac{10}{3}$.

Therefore the equation of the tangent line to that ellipse at the point $(4, -1)$ is $y + 1 = -\frac{10}{3}(x - 4)$.

3. (*15 points*) Find the linearization $L(x)$ of the function $g(x) = xf(x^2)$ at $x = 2$ given the following information $f(2) = -1, f'(2) = 4, f(4) = 5, f'(4) = -2$.

Solution: The linearization of a function $g(x)$ at $x = a$ is given by $L(x) = g(a) + g'(a)(x - a)$.

We need to compute $g(2)$ and $g'(2)$.

To evaluate $g(2)$, we substitute $x = 2$: $g(2) = (2)f(2^2) = 2 \cdot f(4) = 2 \cdot 5 = 10$.

Then we find $g'(x)$: $g'(x) = (x)'f(x^2) + x(f(x^2))' = (1)f(x^2) + x(f'(x^2) \cdot 2x) = f(x^2) + 2x^2 f'(x^2)$.

If we evaluate at $x = 2$ we get $g'(2) = f(4) + 2(2)^2 f'(2^2) = 5 + 8(-2) = -11$.

Therefore the linearization is $L(x) = 10 - 11(x - 2)$.

4. (*20 points*) The radius r of a cylinder with base and lid is *increasing* at a rate of 2 cm/s. At the same time its height h is *decreasing* at a rate of 5 cm/s. At what rate is the area increasing (or decreasing) when the radius is 5 cm and the height is 10 cm? [The area of a cylinder with base and lid, of radius r and height h is given by $A = 2\pi rh + 2\pi r^2$.]

Solution: We need to find the rate of change of the surface area, that is, $\dfrac{dA}{dt}$, when $r = 5$cm and $h = 10$cm.

We are given the following rates of change:

- The radius r is increasing at a rate of $\dfrac{dr}{dt} = 2$ cm/s,
- The height h is decreasing at a rate of $\dfrac{dh}{dt} = -5$ cm/s.

To find the rate of change of the surface area, we differentiate A with respect to time t:

$$\frac{dA}{dt} = \frac{d}{dt}\left(2\pi r^2 + 2\pi rh\right)$$

Using the chain rule, using the fact that r and h are functions of t, we differentiate each term to get

$$\frac{dA}{dt} = 2\pi \cdot 2r \cdot \frac{dr}{dt} + 2\pi\left(r \cdot \frac{dh}{dt} + h \cdot \frac{dr}{dt}\right)$$

Now substitute the given values: $r = 5$ cm, $h = 10$ cm, $\dfrac{dr}{dt} = 2$ cm/s, and $\dfrac{dh}{dt} = -5$ cm/s to get

$$\frac{dA}{dt} = 4\pi(5)(2) + 2\pi\left(5 \cdot (-5) + 10 \cdot 2\right)$$

Finally, simplify the expression and solve the equation to get

$$\frac{dA}{dt} = 40\pi + 2\pi\left(-25 + 20\right) = 40\pi + 2\pi(-5) = 30\pi$$

Thus, the rate at which the surface area is changing when the radius is 5 cm and the height is 10 cm is 30π cm^2/s. This means that the surface area is increasing at a rate of 30π cm^2/s.

5. (*8 points*) Find the following limits:

(a) $\lim_{x \to 0} \frac{\sin(4x)}{\sin(5x)}$ (b) $\lim_{x \to \infty} \frac{\sinh(4x)}{\cosh(5x)}$

Solution:

(a) We know that $\lim_{t \to 0} \frac{\sin(t)}{t} = 1$. Therefore, $\lim_{x \to 0} \frac{\sin(4x)}{4x} = 1$ and $\lim_{x \to 0} \frac{\sin(5x)}{5x} = 1$, and also $\lim_{x \to 0} \frac{5x}{\sin(5x)} = \lim_{x \to 0} \frac{1}{\frac{\sin(5x)}{5x}} = \frac{1}{1} = 1$.

Let us write the limit we want using the limits we know. We have

$$\lim_{x \to 0} \frac{\sin(4x)}{\sin(5x)} = \lim_{x \to 0} \frac{\sin(4x)}{4x} \frac{5x}{\sin(5x)} \frac{4x}{5x} = \frac{4}{5},$$

which gives the answer.

(b) $\lim_{x \to \infty} \frac{\sinh(4x)}{\cosh(5x)}$. Recall the exact definitions of hyperbolic sine and cosine:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

Substitute these definitions into the expression:

$$\frac{\sinh(4x)}{\cosh(5x)} = \frac{\frac{e^{4x} - e^{-4x}}{2}}{\frac{e^{5x} + e^{-5x}}{2}}$$

Simplifying, we get:

$$\frac{\sinh(4x)}{\cosh(5x)} = \frac{e^{4x} - e^{-4x}}{e^{5x} + e^{-5x}}$$

As $x \to \infty$, $e^{4x} \to \infty$, $e^{5x} \to \infty$, $e^{-4x} \to 0$, $e^{-5x} \to 0$. Thus, let us divide numerator and denominator by the 'strongest' term in the denominator to get

$$\lim_{x \to \infty} \frac{\sinh(4x)}{\cosh(5x)} = \lim_{x \to \infty} \frac{e^{4x} - e^{-4x}}{e^{5x} + e^{-5x}} = \lim_{x \to \infty} \frac{\frac{e^{4x} - e^{-4x}}{e^{5x}}}{\frac{e^{5x} + e^{-5x}}{e^{5x}}} = \lim_{x \to \infty} \frac{e^{-x} - e^{-9x}}{1 + e^{-10x}} = \frac{0 - 0}{1 - 0} = 0.$$

6. (*15 points*) Suppose that f and g are differentiable functions such that $f(g(x)) = x$. We only know that $g(3) = 5$ and that $f'(5) = -12$. Find $g'(3)$.

Solution:

Differentiate both sides of the equation $f(g(x)) = x$, using the chain rule, to get $f'(g(x)) \cdot g'(x) = 1$.

Substitute $x = 3$, $g(3) = 5$, and $f'(5) = -12$ to get $f'(g(3)) \cdot g'(3) = 1$, or $f'(5) \cdot g'(3) = 1$, or $(-12) \cdot g'(3) = 1$

Therefore, $g'(3) = -\frac{1}{12}$