PARABOLAS, LINES AND PARABOLAS

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Define the *parabola* as the locus of points that they are equidistant from a line (the *directrix*) and a point (the *focus*).
Define the vertex of the parabola as the midpoint of the segment perpendicular to the directrix starting at the focus.
Examples:

(1) Find the equation of the parabola with directrix $l : y = -2$ and focus at $F : (0, 2)$.

Solution. Let (x, y) be any point. Then the distance from l is $|y+2|$ while the distance for F is $\sqrt{x^2 + (y-2)^2}$. So a point with coordinates (x, y) is in the parabola, if and only if, the following equation holds:

$$\sqrt{x^2 + (y-2)^2} = |y+2|$$

To get the equation in a simpler form lets square both sides:

$$x^2 + (y-2)^2 = (y+2)^2$$

Let us expand:

$$x^2 + y^2 - 4y + 4 = y^2 + 4y + 4$$

Which upon transfering everything to the LHS becomes:

$$x^2 - 8y = 0$$

Finally, solving for y gives us the more familiar form:

$$y = \frac{x^2}{8}$$

□

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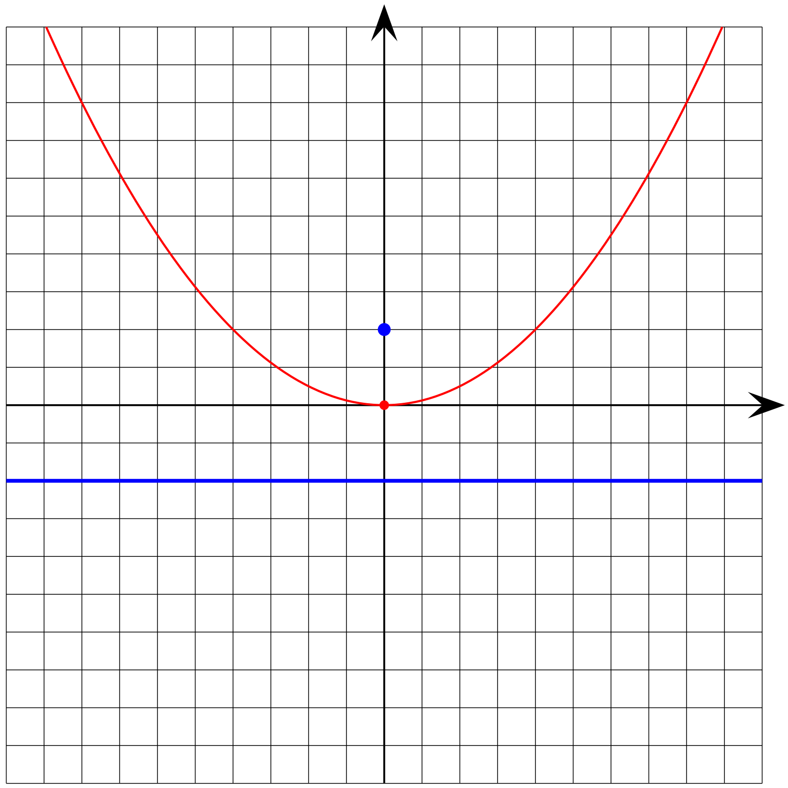


(2) How about the parabola $y = x^2$? It turns out that this has focus $(0, \frac{1}{4})$ and directrix $y = \frac{1}{4}$.

(3) Find the equation of the parabola with focus $(0, -1)$ and directrix $y = 1$.

Solution. We must have:

$$\sqrt{x^2 + (y+1)^2} = |y - 1|$$

We square both sides:

$$x^2 + (y+1)^2 = |y - 1|^2$$

Now we expand:

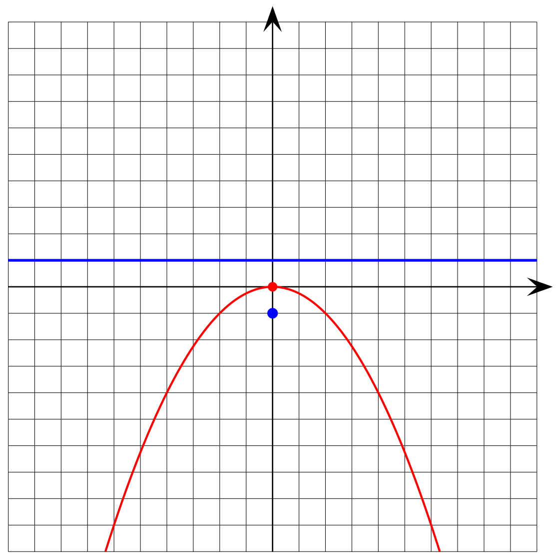
$$x^2 + y^2 + 2y + 1 = y^2 - 2y + 1$$

After transfering everything to the LHS and simplifying we get:

$$x^2 + 4y = 0$$

And finally solving for y we get:

$$y = -\frac{x^2}{4}$$



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(4) Find the equation of the parabola with focus $F:(0,q)$ and directrix $y=c$.

Solution. The point (x,y) has to satisfy:

$$x^2+(y-q)^2=(y-c)^2$$

After expanding we get:

$$x^2+y^2-2qy+q^2=y^2-2cy+c^2$$

Which after the usual simplifications gives:

$$x^2+2cy-2qy+q^2-c^2=0$$

Or after collectng terms with y:

$$x^2+2(c-q)y+q^2-c^2=0$$

Solving for y gives:

$$y=-\frac{x^2+q^2-c^2}{2(c-q)}$$

We further simplify this as follows:

$$y=\frac{x^2}{2(q-c)}+\frac{q^2-c^2}{2(q-c)}$$

So finally we get:

$$y=\frac{x^2}{2(q-c)}+\frac{q+c}{2}$$

□

(5) Find the equation of the parabola with focus $F:(p,q)$ and directrix $l:y=c$.

Solution. A point in this parabola will have to satisfy the equation:

$$\sqrt{(x-p)^2+(y-q)^2} = |y-c|$$

After squaring both sides, expanding, and solving for y we get:

$$y = \frac{x^2-2px+p^2+q^2-c^2}{2q-2c}$$

We can further write the RHS of the equation as follows:

$$\begin{aligned}\frac{x^2-2px+p^2+q^2-c^2}{2q-2c} &= \frac{x^2-2px+p^2}{2(q-c)} + \frac{q^2-c^2}{2(q-c)}\\ &= \frac{(x-p)^2}{2(q-c)} + \frac{(q-c)(q+c)}{2(q-c)}\\ &= \frac{(x-p)^2}{2q-2c} + \frac{q+c}{2}\end{aligned}$$

So that the equation of the parabola is:

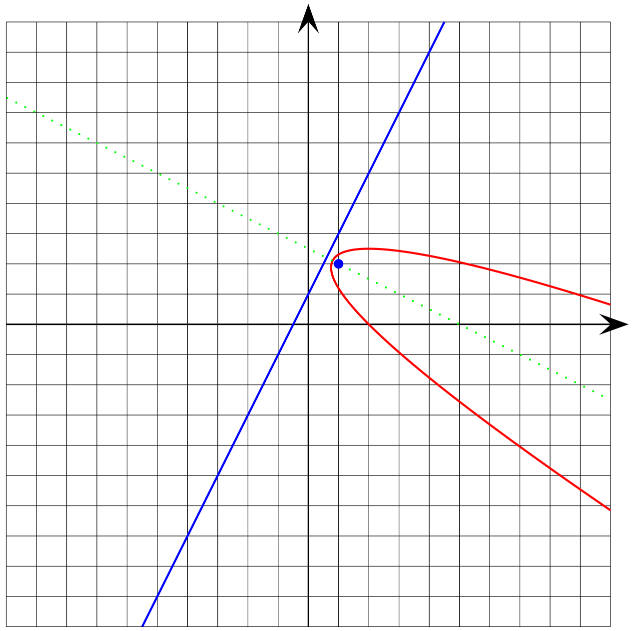
$$y = \frac{(x-p)^2}{2q-2c} + \frac{q+c}{2}$$

From this form of the equation we can see that the vertex is at $(p, \dfrac{q+c}{2})$, as expected. □

(6) Find the equation of the parabola with focus $(1,0)$ and directrix $x=-1$.
(7) Find the equation of the parabola with focus $(1,2)$ and directrix $y=2x+1$

Answer.

$$x^2+4y^2+4xy-14x-18y+24=0$$



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Proposition: The equation $y = ax^2$ describes a parabola with

- focus $(0, \frac{1}{4a})$, directrix $y = -\frac{1}{4a}$, and vertex at $(0, 0)$, if $a > 0$.
- focus $(0, -\frac{1}{4a})$, directrix $y = \frac{1}{4a}$, and vertex at $(0, 0)$, if $a < 0$.

Examples:

(1) Find a line that touches the parabola $y = x^2$ at the point $(1, 1)$.

Solution. The equation of a non-vertical line that passes through $(1, 1)$ is $y = mx - m + 1$. At a common point of this line with the parabola we have:

$$x^2 = mx - m + 1$$

or equivalently

$$x^2 - mx + m - 1 = 0$$

Now the discriminant of this quadratic equation is

$$(-m)^2 - 4 \cdot 1 \cdot (m - 1) = m^2 - 4m + 4 = (m - 2)^2$$

In order for the quadratic equation to have only one (double) solution we need the disciminant to be 0. This happens exactly when $m = 2$. So there is only one common point when $m = 2$. In that case the line has equation:

$$y = 2x - 1$$

□

(2) Find a line that touches the parabola $y = x^2$ at the point $(3, 9)$.

Solution. An non-vertical line passing through $(3,9)$ has equation $y = mx - 3m + 9$, for some m. To find common points with the parabola we have to solve the equation:

$$x^2 - mx + 3m - 9 = 0$$

The discriminant of this equation is

$$m^2 - 4(3m - 9) = m^2 - 12m - 36 = (m - 6)^2$$

So the discriminant is 0 when $m = 6$, and the equation of the tangent line is

$$y = 6x - 9$$

□

(3) In general the tangent at the point (p, p^2) will have slope $2p$.

Proof. In general, a line that passes through (p, p^2) has equation $y = mx - pm + p^2$. The common points of such a line with the parabola $y = x^2$ have x-coordinates that satisfy the equation:

$$x^2 - mx + pm - p^2 = 0$$

The discriminant of this equation is

$$m^2 - 4pm - 4p^2 = (m - 2p)^2$$

So there is a double solution exactly when $m = 2p$. The tangent line is

$$y = 2px - p^2$$

□

Exercise Extra Credit: Consider the parabola: $y = -2x^2$.

(1) Find the tangent at the point $(1, -2)$.
(2) Find the tangent at the point $(-1, -2)$
(3) What is the slope at the tangent $(p, -2p^2)$?

Completing the square: Find the focus, directrix, vertex and axis of symmetry of the following parabolas:

(1) $y = x^2 - 6x + 8$
(2) $y + x^2 - 4x + 9 = 0$
(3) $x - y^2 - 2y + 2 = 0$