POINTS, LINES, DISTANCES

NIKOS APOSTOLAKIS

Examples/Exercises:

(1) Find the equation of the line that passes through $(4, 5)$, $(4, -2)$
(2) Find the equation of the line that passes through the points $(1, 2)$, $(3, 4)$.
(3) Find the equation of the line that is parallel to $3x - 4y = 5$ and passes through $(1, 3)$.
(4) Find the equation of the line that is perpendicular to $y = 2x + 3$ and passes through $(0, -2)$
(5) What are the intercepts of $ax + by - c = 0$? What is its slope?
(6) Consider the line $5x - 7y = 21$. For each of the following points determine whether they are on the line, below the line, or above the line.
   (a) $(1, 2)$
   (b) $(7, 8)$
   (c) $(4, -\frac{1}{7})$
   (d) $(\frac{14}{5}, -1)$
   (e) $(-8, -9)$
   (f) $(-9, 6)$
   (g) $(10, 20))$
(7) Of those points in the previous exercise that are not on the line which are to the left and which are to the right of the line?

Recall: When two lines intersect, when parallel, when two equations describe the same line, when they are perpendicular. Systems of equations.
Fact: The slope-intercept form of the equation of a line is unique. in other words two different lines have different equations.
Fact: Two linear equations in standar form represent the same line if and only if one is a multiple of the other.
Recall: Equal slope means parallel. If the product of the slopes is -1 then the lines meet at a right angle.
Examples:

(1) The equations $2x - 3y = 4$ and $4x - 6y = 8$ represent the same line.
(2) The equations $3x + 5y = 6$ and $x + \frac{5}{3}y = 2$ represent the same line.
(3) The equations $-4x + 5y = 1$ and $-4x + 5y = 3$ represent parallel lines.
(4) The equations $2x + y = 1$ and $x - 2y = 3$ represent perpendicular lines.
(5) The equations $3x + 2y = 4$ and $2x - y = 7$ represent intersecting lines that are not parallel.

Project <u>Extra Credit:</u> Prove the assertions above. That is:

(1) Show that two *different* lines with the same slope never meet, by showing that the corresponding system has no solutions.
(2) Show that two lines with different slope always meet, by showing that the corresponding system can always be solved.

Discuss: Analytic versus synthetic geometry. Using algebra to solve geometric problems.

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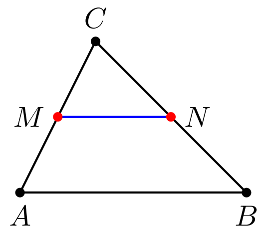
Example: In any triangle, the segment that joins the midpoints of any two sides is parallel to the third side.



Proof. Here is how we can prove this using coordinates. Let $A:(r,s)$, $B:(v,w)$, and $C:(p,q)$. Then the midpoint M of AC has coordinates $M:\left(\frac{r+p}{2},\frac{s+q}{2}\right)$ and the midpoint N of BC has coordinates $N:\left(\frac{v+p}{2},\frac{w+q}{2}\right)$. Then the slope of the line MN is

$$\frac{\frac{w+q}{2}-\frac{s+q}{2}}{\frac{v+p}{2}-\frac{r+p}{2}}=\frac{w+q-(s+q)}{v+p-(r+p)}=\frac{w-s}{v-r}$$

Which is also the slope of AB. Therefore $MN \parallel AB$, as needed. ☐

This was not that hard, but by *carefully choosing the coordinates* we can make it even easier:

Alternative proof. We are free to chose our coordinate system. If two lines are parallel they will be so no matter the coordinate system we choose. So if we choose a convenient coordinate system to do our calculations and MN has the same slope as AB in those coordinates then $MN \parallel AB$. So lets choose coordinates so that: $A:(0,0)$, $B:(2,0)$, and $C:(p,q)$. Then the slope of AB is 0. The midpoints have coordinates $M:\left(\frac{p}{2},\frac{q}{2}\right)$ and $N:\left(\frac{p+2}{2},\frac{q}{2}\right)$ so that the slope of MN is also 0. ☐

Exercise Extra Credit: Prove that the midpoints of the sides of any quadrilateral form a parallelogram.
Recall: distance formula, pythagorean theorem.
Discuss: What is a *geometric locus* and how to find it.
Define: The set of points in the plane that satisfy a certain condition is called a *gemetric locus.*
Example: The geometric locus of points that are distance 5 apart from the point $(0,-1)$ is a circle. It's equation is

$$x^2+(y+1)^2=25$$

Example: Find the locus of points equidistant from two given points.

(1) Points $(0,3)$, $(0,5)$
(2) Points $(-1,0)$, $(2,0)$
(3) Points $(2,4)$, $(6,4)$
(4) Points $(1,2)$, $(-1,5)$

Proposition: The locus of points that are in equal distance from two given points P, Q is the line that passes through the midpoint of the segment PQ and is perpendicular to PQ.

Proof. We are free to choose our coordinate system! So let's choose a coordinate system that has the point P in its origin and the point Q in the x–axis, at the point $(1,0)$. ☐

Example: Use the above proposition to compute an equation for the locus of points equidistant from $(-2, 3)$ and $(1, 4)$.
Exercises: Use the above proposition to compute an equation for the locus of points equidistant from the points with coordinates:

(1) $(-2, 1)$ and $(3, -4)$
(2) $(-4, -3)$ and $(-4, 5)$
(3) $(1, 2)$ and $(2, 1)$
(4) $(-3, 4)$ and $(4, -3)$
(5) Extra Credit: (p, q) and (q, p)

Define: Distance of a point P from a line l as the smallest of all the distances of P from points of l.
Problem: How can we find it?
Observation: Pythagorean theorem implies that the perpendicular line is the path of shortest distance. Indeed the hypotenuse is always the largest of the three sides.

Proof.

$$\begin{aligned} a^2 + b^2 = c^2 &\Leftrightarrow c^2 - a^2 = b^2 \\ &\Leftrightarrow (c - a)(c + a) = b^2 \\ &\Leftrightarrow c - a = \frac{b^2}{c + a} \end{aligned}$$

So $c - a$ is positive and therefore $c > a$. □

So to find the distance of P from l we can follow the following proceedure:

(1) Find the equation of the line $l^\perp$ that passes through P and is perpendicular to l.
(2) Find the intersection point Q of l and $l^\perp$. Usually Q is called the "foot" of the perpendicular.
(3) Find the distance of P and Q.

Examples: We give some examples of this procedure:

(1) Consider the line $l : 2x - 3y = 6$ and the point $P : (-1, 3)$. Find the (shortest) distance from P to l.

Solution. The line $l^\perp$ that is perpendicular to l and passes through P, has equation: $l^\perp : 2y + 3x = 3$.[1]

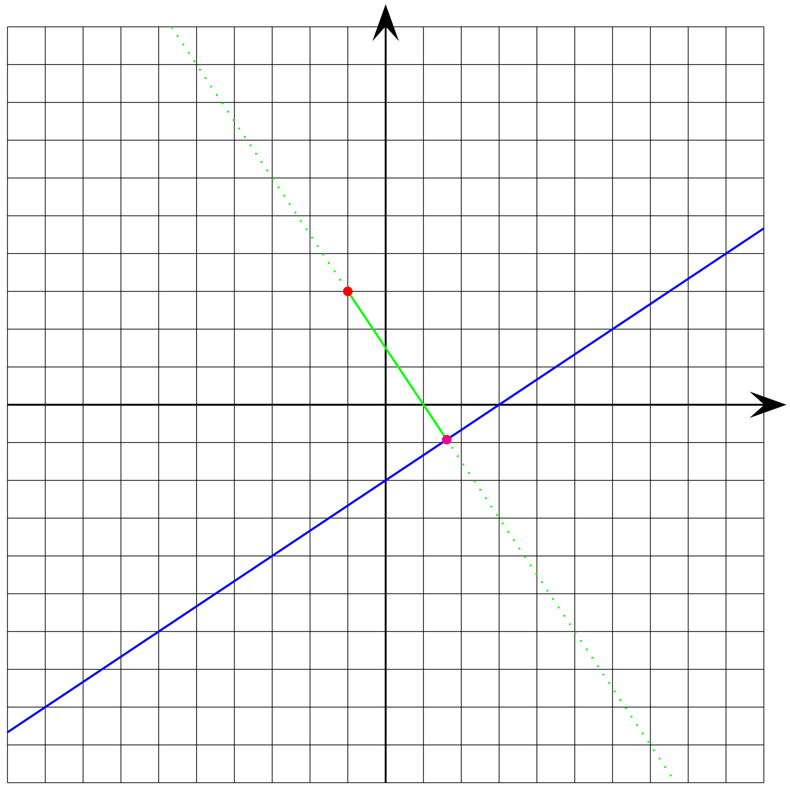
Now we find the "foot" of the perpendicular line. To do this we solve the system:

$$\begin{cases} 2x - 3y &= 6 \\ 3x + 2y &= 3 \end{cases}$$

We find that the foot is $Q : \left(\frac{21}{13}, -\frac{12}{13}\right)$. It follows that the shortest distance of P and l is the distance of P and Q. So the distance is

$$\sqrt{\left(\frac{21}{13} + 1\right)^2 + \left(-\frac{12}{13} - 3\right)^2} = \frac{17\sqrt{13}}{13}$$

[1] Why? Do the calculations!



□

(2) Find the distance of the point $(-2,-5)$ from the line $y = 2x - 1$.
(3) Find the distance of a generic point (p, q) from the line $x - y - 5 = 0$.

Solution. A line that is perpendicular to l will have equation $x + y - c = 0$ for some real number c. Now since it passes through (p, q) after substituting we get a true equation. So we must have:

$$p + q - c = 0$$

or in other words

$$c = p + q$$

The equation of the perpendicular line is therefore:

$$x + y - p - q = 0$$

Solving the system gives:

$$x = \frac{p + q + 5}{2}, \quad y = \frac{p + q - 5}{2}$$

Calculating Δy and Δx

$$\Delta x = \frac{-p + q + 5}{2}, \quad \Delta y = -\frac{-p + q + 5}{2}$$

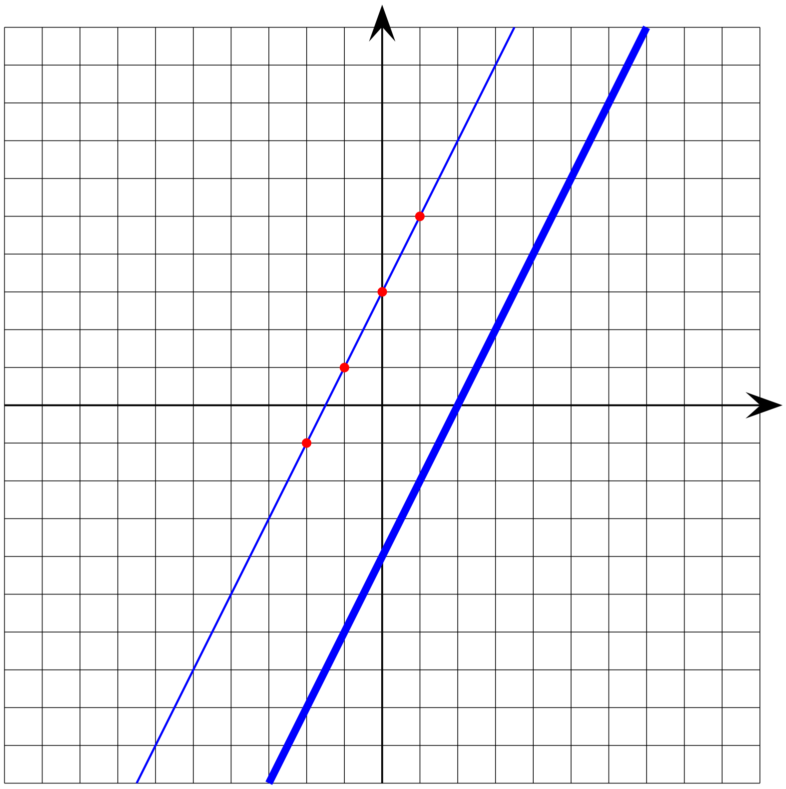
So the square of the distance is

$$d^2 = \frac{(p - q - 5)^2}{2}$$

□

(4) Find the distance of the generic point (p, q) from the line $y = x$.

(5) In this example we consider a line, say $l : 2x - y = 4$. Then we know that the lines with the same "variable part", i.e. lines of the form $2x - y = c$ are parallel to l. So let's take one of them $2x - y = 3$, and chose four points in it, say, $(-2, -1)$, $(-1, 1)$, $(0, 3)$, $(1, 5)$. Let's calculate the distance of all these points from l, if our calculations are correct all these distances should be equal.



Exercises:

(1) Find the distance of the point $(1, 2)$ from the line $x = 3$
(2) Find the distance of the point $(-3, -5)$ from the line $3x - 4y = 6$.
(3) Find the distance of the point $(0, 0)$ from the line $ax + by = c$.
(4) Find the distance of a point (p, q) from the line $2x + y = 3$.
(5) Find the distance of a point (p, q) from the line $3x - 2y = 7$.
(6) Let $P : (p, q)$ and $S : (q, p)$ be two points whose coordinates have been interchanged. Prove that
   (a) The line passing trough P and S is perpendicular to the diagonal line $y = x$.
   (b) P and S are in the same distance from $y = x$.
(7) <u>Extra Credit:</u> Find a formula for the distance of a generic point (p, q) from a general line $ax + by = c$.