CIRCLES, LINES AND CIRCLES, TWO CIRCLES

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Recall: Definition of a circle. Equation of a circle. How to find the center and radius of a circle by completing the square. A circle has an inside and an outside.
Exercises:

(1) Find the equation of the circle with center $(1, -2)$ and radius 5.
(2) Find the equation of the circle with center $(0, 3)$ and passing through the point $(4, 5)$.
(3) For each of the folloing points decide their relative position (i.e. whether they are inside, on, or outside) with respect to the circle $x^2 + y^2 = 4$.
  (a) $(-1, \sqrt{3})$
  (b) $(1, 1)$
  (c) $(2, 0)$
  (d) $(-3, 1)$
  (e) $(\sqrt{2}, -\sqrt{2})$
(4) Find the center and the radius of the circle with equation $x^2 + y^2 - 4x + 10y = -20$
(5) Ditto for the circle $4x^2 + 4y^2 - 4x - 40y + 85 = 0$
(6) How about the equation $2x^2 + y^2 - 4x + 4y + 5 = 0$. Does it represent a circle? If not do you know what shape it does represent?

Examples: Find the common points of the following circle and line.

(1) Circle $x^2 + y^2 = 1$ and line $x = \frac{1}{2}$
(2) Circle $x^2 + y^2 = 9$ and line $y = 2$
(3) Circle $x^2 + y^2 = 4$ and line $x = -2$
(4) Circle $x^2 + (y + 3)^2 = 5$ and line $y = 2x - 1$.
(5) Circle $x^2 + y^2 = 1$ and line $x = \frac{1}{2}$.
(6) Circle $(x - 2)^2 + (y + 1)^2 = 4$ and line $2x + y = 10$.
(7) Circle $(x - 1)^2 + y^2 = 1$ and line $2x + y = 10$.
(8) Circle $(x - 1)^2 + y^2 = 5$ and line $y = -\frac{x}{2} + 3$

Proposition: Given a circle and a line there are three possibilities: they meet at two points, they touch at one point or they have no points in common.

Proof. The equation of the line can be solved for one of the variables. Substituting in the equation of the circle gives a quadratic equation in one variable. Such equation has zero, one, or two real solutions (depending on the sign of the discriminant). □

Define: Tangent of a circle at a point. A line that touches the circle.
Question: How can we find the tangent to a circle?
Let's see some **Examples:**

(1) Find the tangent to the circle $c\colon x^2 + y^2 = 8$ at the point $P\colon (-2, 2)$.

Date: 31 January 2010.

Solution. We first check the vertical line that passes through the given point, i.e. the line $x = -2$. Substituting into the equation of c gives:

$$\begin{aligned}(-2)^2 + y^2 = 8 &\iff 4 + y^2 = 8\\ &\iff y^2 = 4\\ &\iff y = \pm 2\end{aligned}$$

So this line has two common points with the circle.

Next we check non-vertical lines. Such a line will have equation of the form $y = mx + b$. In order for this line to contain the point P we must have

$$2 = -2m + b \iff b = 2m + 2$$

So the equation of a non-vertical line that passes through P has the form

$$l\colon\ y = mx + 2m + 2$$

Substituting this in to the equation of c, we see that the x–coordinate of the common points of l and c must satisfy the equation:

$$\begin{aligned}x^2 + (mx + 2m + 2)^2 = 8 &\iff x^2 + m^2x^2 + 4m^2x + 4mx + 4m^2 + 8m + 4 = 8\\ &\iff \left(m^2 + 1\right)x^2 + \left(4m^2 + 4m\right)x + 4m^2 + 8m - 4 = 0\end{aligned}$$

In order for this equation to have only one (double) solution we need its discriminant to be 0. So we need:

$$\begin{aligned}(4m^2 + 4m)^2 - 4(m^2 + 1)(4m^2 + 8m - 4) = 0 &\iff 16m^4 + 32m^3 + 16m^2 - 16m^4 - 32m^3\\ &\qquad + 16m^2 - 16m^2 - 32m + 16 = 0\\ &\iff 16m^2 - 32m + 16 = 0\\ &\iff 16(m^2 - 2m + 1) = 0\\ &\iff m^2 - 2m + 1 = 0\\ &\iff (m - 1)^2 = 0\\ &\iff m - 1 = 0\\ &\iff m = 1\end{aligned}$$

So the equation of the tangent line has to be:

$$y = x + 4$$

□

(2) Find the tangent to the circle $(x - 2)^2 + (y - 3)^2 = 5$ at the point $(1, 1)$.

Solution. A non-vertical line that passes through $(1, 1)$ has equation $y = mx - m + 1$. Substituting in the equation of the circle and expanding gives:

$$(m^2 + 1)x^2 - (4 + 4m + 2m^2)x + (m^2 + 4m + 3)$$

The discriminant of this equation turns out to be:

$$4(2m + 1)^2$$

In order to have a single solution we need to have the discriminant 0, so we need:

$$m = -\frac{1}{2}$$

In sum, the equation is:

$$y = -\frac{x}{2} + \frac{3}{2}$$

□

(3) Find the tangent to the circle $x^2 + y^2 = 1$ at the point $(1, 0)$.

Proposition: The tangent of a circle at a point is perpendicular to the radius of the circle at the same point.

Proof. We can choose the coordinate system so that the origin is at the center of the circle, the circle has radius 1 and the point is $(0, 1)$. According to example 2) above the tangent is $x = 1$ which is perpendicular to the radius.

Now whether two lines are perpendicular does not depend on the coordinate system chosen so it follows that the tangent line of *any* circle at *any* point is perpendicular to the radius. □

Exercises: Use the above proposition to find the tangent of the given circle at the given point.

(1) Cicrle $(x + 1)^2 + (y - 2)^2 = 10$, point $(2, 3)$.
(2) Cicrle $x^2 + (y - 1)^2 = 2$, point $(-2, 1)$.
(3) Circle $(x + 1)^2 + (x + 2)^2 = 40$ point $(-3, 4)$.
(4) Circe $x^2 + y^2 = 1$ point $(\frac{2\sqrt{2}}{2}, \frac{-2\sqrt{2}}{2})$.

Project: What are the relative positions of two circles? Give an example for each of the five possibilities shown in Figure 1. Are ther other posibilities?

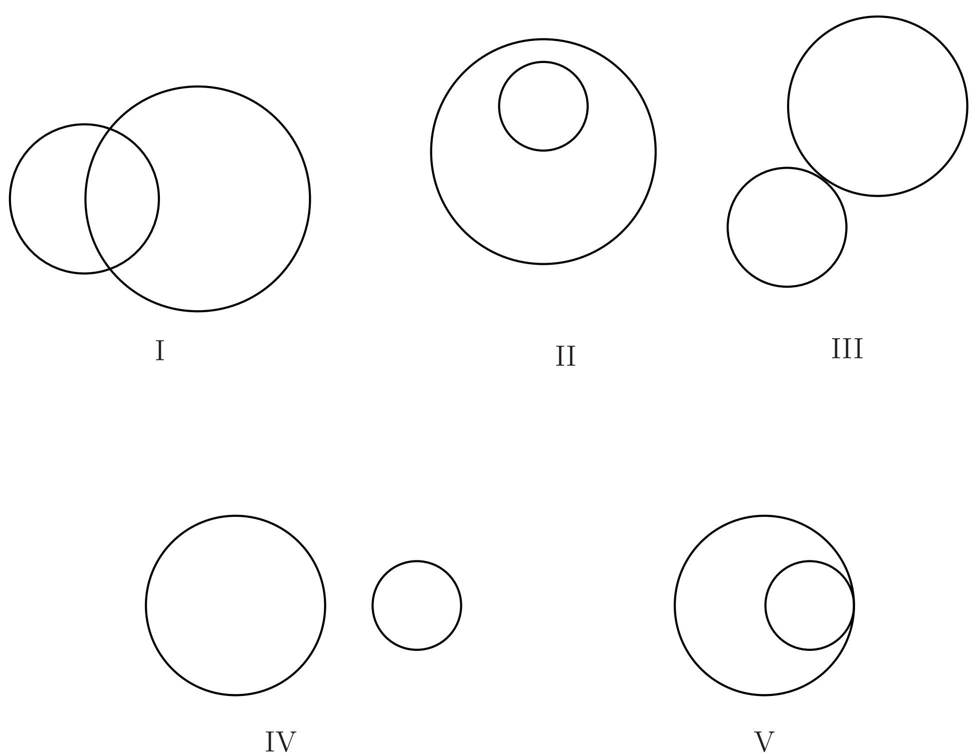

FIGURE 1. Five possibilities for two circles

Exercises:

(1) Find the common points of the two circles:
(a) $x^2 + y^2 = 4$,

   (b) $(x-3)^2+y^2=9$
   (c) $(x-1)^2+(y+1)^2=25$, $(x-1)^2+(y-1)^2=4$
   (d) $(x-1)^2+(y+2)^2=4$, $(x-4)^2+(y-2)^2=9$

(2) A circle C_1 is centered at $(6,8)$ and is tangent to the circle $C_2 : x^2+y^2=25$. Find the radius of C_1 and the point of contact of C_1 and C_2.

(3) Extra Credit: Find the locus of the centers of all circles that have radius 2 and are tangent to the circle $x^2+(y-1)^2=9$.