MATH 30 - Precalculus. Homework 5. Due Th. 03/21/2019. Professor Luis Fernández

SOLUTION

Write your answers in other sheets and **STAPLE this one to your other sheets.**

1. Do exercises 2, 4 and 6 from Exercise Set 2.5 of the book (page 335 of the 3rd edition).

Solution: We have to list all possible rational zeros for the following functions:

Ex. 2: $f(x) = x^3 + 3x^2 - 6x - 8$. The possible rational roots have the form $\frac{p}{q}$ where p is a factor of 8 and q is a factor of 1. Thus the possibilities for p are $\pm1, \pm2, \pm4, \pm8$ and the possibilities for q are ±1. Therefore the list of all possible rational roots is

$$\pm1, \pm2, \pm4, \pm8.$$

Ex. 4: $f(x) = 2x^4 + 3x^3 - 11x^2 - 9x + 15$. The possible rational roots have the form $\frac{p}{q}$ where p is a factor of 15 and q is a factor of 2. Thus the possibilities for p are $\pm1, \pm3, \pm5, \pm15$ and the possibilities for q are $\pm1, \pm2$. Therefore the list of all possible rational roots is

$$\pm1, \pm3, \pm5, \pm15, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2}.$$

Ex. 6: $f(x) = 3x^4 - 11x^3 - 3x^2 - 6x + 8$. The possible rational roots have the form $\frac{p}{q}$ where p is a factor of 8 and q is a factor of 3. Thus the possibilities for p are $\pm1, \pm2, \pm4, \pm8$ and the possibilities for q are $\pm1, \pm3$. Therefore the list of all possible rational roots is

$$\pm1, \pm2, \pm4, \pm8, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}.$$

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2. Do exercises 12, 14, 22 and 24 from Exercise Set 2.5 of the book (page 336 of the 3rd edition).

Solution:

Ex. 12: The possible rational zeros of $f(x) = 2x^3 - 5x^2 + x + 2$ are ±1, ±2 and $\pm\frac{1}{2}$. Now we do synthetic division to test them:

$$\begin{array}{r|rrrr} & 2 & -5 & 1 & 2 \\ 1 & & 2 & -3 & -2 \\ \hline & 2 & -3 & -2 & 0 \\ 2 & & 4 & 2 & \\ \hline & 2 & 1 & 0 & \end{array}$$

Therefore $f(x) = 2x^3 - 5x^2 + x + 2 = (x-1)(x-2)(2x+1)$. So the zeros of f are 1, 2 and $-\frac{1}{2}$.

($-\frac{1}{2}$ is the solution of writing the last factor of $f(x)$ equal to 0, i.e. $(2x+1) = 0$.)

Ex. 14: The possible rational zeros of $f(x) = 2x^3 + x^2 - 3x + 1$ are ±1, $\pm\frac{1}{2}$. Now we do synthetic division to test them. Check that 1 and -1 are not roots. However, $\frac{1}{2}$ is:

$$\begin{array}{r|rrrr} & 2 & 1 & -3 & 1 \\ 1/2 & & 1 & 1 & -1 \\ \hline & 2 & 2 & -2 & 0 \end{array}$$

Therefore $f(x) = 2x^3 + x^2 - 3x + 1 = (x - \frac{1}{2})(2x^2 + 2x - 2)$.

To find the other two zeros, use the quadratic formula:

$x = \dfrac{-2 \pm \sqrt{4+16}}{4} = \dfrac{-2 \pm \sqrt{20}}{4} = \dfrac{-1 \pm \sqrt{5}}{2}$.

Thus, the zeros of f are $\frac{1}{2}$, $\dfrac{-1+\sqrt{5}}{2}$ and $\dfrac{-1-\sqrt{5}}{2}$.

Ex. 22: The possible rational zeros of $f(x) = 2x^3 - 5x^2 - 6x + 4$ are ±1, ±2, ±4, $\pm\frac{1}{2}$. Now we do synthetic division to test them. Check that ±1, ±2, ±4 are not roots. However, $\frac{1}{2}$ is:

$$\begin{array}{r|rrrr} & 2 & -5 & -6 & 4 \\ 1/2 & & 1 & -2 & -4 \\ \hline & 2 & -4 & -8 & 0 \end{array}$$

Therefore $f(x) = 2x^3 - 5x^2 - 6x + 4 = (x - \frac{1}{2})(2x^2 - 4x - 8)$.

The final answer is: the roots of the equation $2x^3 - 5x^2 - 6x + 4 = 0$ are $\frac{1}{2}$, $-1 + \sqrt{5}$ and $-1 - \sqrt{5}$.

Ex. 24: The possible rational zeros of $f(x) = x^4 - 2x^2 - 16x - 15$ are ± 1, ± 3, ± 5, ± 15. Now we do synthetic division to test them. Check that 1 is not a root. However, -1 is:

$$\begin{array}{r|rrrrr}
 & 1 & 0 & -2 & -16 & -15 \\
-1 & & -1 & 1 & 1 & 15 \\
\hline
 & 1 & -1 & -1 & -15 & 0 \\
3 & & 3 & 6 & 15 & \\
\hline
 & 1 & 2 & 5 & 0 &
\end{array}$$

Therefore we have to solve $x^4 - 2x^2 - 16x - 15 = (x+1)(x-3)(x^2+2x+5) = 0$. So it only remains to solve $x^2 + 2x + 5 = 0$, for which you can use the quadratic formula. The final answer is: The roots of the equation $x^4 - 2x^2 - 16x - 15 = 0$ are $-1, 3, -1-2i, -1+2i$.

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3. Solve the following polynomial equations. [How? We did several examples in class; or look at examples 3, 4 and 5 of the book, section 2.5.]

a) $x^3 - 4x^2 - 7x + 10 = 0$

b) $3x^3 - 8x^2 - 8x + 8 = 0$

c) $x^4 + 3x^3 - 20x^2 + 24x - 8 = 0$

d) $x^4 - x^3 + 2x^2 - 4x - 8 = 0$

Solution:

a) The possible rational solutions of $x^3 - 4x^2 - 7x + 10 = 0$ are ± 1, ± 2, ± 5, ± 10. Now we do synthetic division to test them. Check that 1 is not a root. However, -1 is:

$$\begin{array}{r|rrrr}
 & 1 & -4 & -7 & 10 \\
1 & & 1 & -3 & -10 \\
\hline
 & 1 & -3 & -10 & 0 \\
-2 & & -2 & 10 & \\
\hline
 & 1 & -5 & 0 & \\
5 & & 5 & & \\
\hline
 & 1 & 0 & &
\end{array}$$

Therefore the solutions are 1, -2 and 5.

b) I only write the solutions (proceed as in the previous exercise, or as in exercise 2). They are $\frac{2}{3}$, $1+\sqrt{5}$, $1-\sqrt{5}$.

c) I only write the solutions (proceed as in the previous exercise, or as in exercise 2). They are: 1, 2, $-3-\sqrt{13}$, $-3+\sqrt{13}$.

d) I only write the solutions (proceed as in the previous exercise, or as in exercise 2). They are -1, 2, $2i$, $-2i$.

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4. Use the results of the previous exercise to factor the following polynomials completely.
[NOTE: you DO NOT need to do *any* calculation, only use the *factor theorem*.]

a) $x^3 - 4x^2 - 7x + 10$

b) $3x^3 - 8x^2 - 8x + 8$

c) $x^4 + 3x^3 - 20x^2 + 24x - 8$

d) $x^4 - x^3 + 2x^2 - 4x - 8$

Solution:

a) From the previous exercise, the zeros of $x^3 - 4x^2 - 7x + 10$ are 1, -2 and 5.
Therefore, $x^3 - 4x^2 - 7x + 10 = (x-1)(x+2)(x-5)$

b) From the previous exercise, the zeros of $3x^3 - 8x^2 - 8x + 8$ are $\frac{2}{3}$, $1+\sqrt{5}$, $1-\sqrt{5}$.
Therefore, $3x^3 - 8x^2 - 8x + 8 = (x - \frac{2}{3})(x-(1+\sqrt{5}))(x-(1-\sqrt{5}))$.

c) From the previous exercise, the zeros of $x^4 + 3x^3 - 20x^2 + 24x - 8$ are 1, 2, $-3-\sqrt{13}$, $-3+\sqrt{13}$.
Therefore, $x^4 + 3x^3 - 20x^2 + 24x - 8 = (x-1)(x-2)(x-(-3-\sqrt{13}))(x-(-3+\sqrt{13}))$

d) From the previous exercise, the zeros of $x^4 - x^3 + 2x^2 - 4x - 8$ are -1, 2, $2i$, $-2i$.

Therefore, $x^4 - x^3 + 2x^2 - 4x - 8 = (x+1)(x-2)(x+2i)(x-2i)$.

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5. Solve the equation $(x-1)^2(x-2)(x-3)(x+4) = 0$.
[NOTE: you DO NOT need to do any calculation for this one; use the *factor theorem* to find the solution by just looking at the equation.]
Solution: 1 (with multiplicity two), 2, 3 and -4.

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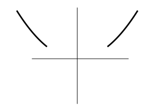
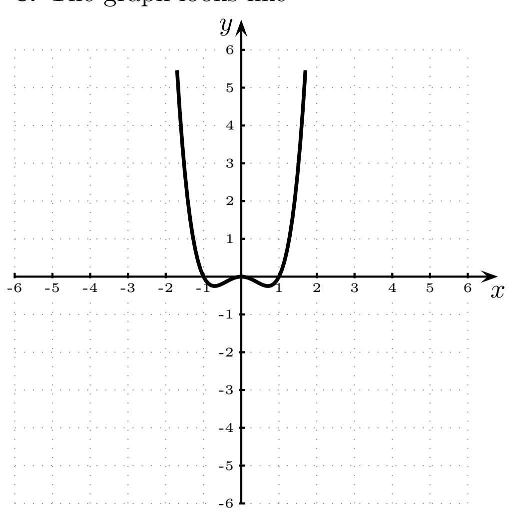
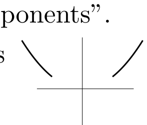
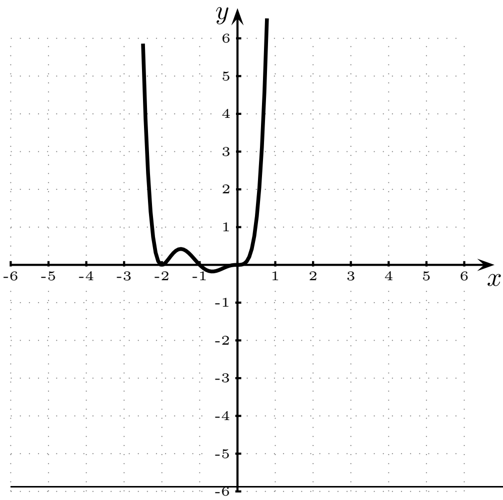
6. Do exercises 44 and 58 from Exercise Set 2.3 of the book. Do not forget to use graph paper to do the graphs.
44. Graph $f(x) = x^4 - x^2$
a. END BEHAVIOUR. Since the leading coefficient is 1 and the degree is even, the end behaviour is: 

b. x-INTERCEPTS. $x^4 - x^2 = x^2(x^2-1) = x^2(x-1)(x+1)$.
Therefore the graph crosses the x-axis ONLY at $x = 1$ and $x = -1$, and touches the x axis and turns around ONLY at $x = 0$ (because 0 has multiplicity 2).
c. y-INTERCEPT. $f(0) = 0$, so the y intercept is $y = 0$.
d. SYMMETRY. Since $f(-x) = (-x)^4 - (-x)^2 = x^4 - x^2 = f(x)$, the function is even, so it is symmetric about the y-axis.
e. The graph looks like



58. Graph $f(x) = x^3(x+2)^2(x+1)$
a. END BEHAVIOUR. If we multiply out the polynomial we get $f(x) = x^6 +$ "more stuff with lower exponents". Therefore the leading coefficient is 1 and the degree is 6, which is even, and therefore the end behaviour is 

b. x-INTERCEPTS. The polynomial is already factored. The roots are 0 (with multiplicity 3), -2 (with multiplicity 2) and -1 (with multiplicity 1).
Therefore the graph crosses the x-axis ONLY at $x = 0$ (where it also flattens out) and $x = -1$, and touches the x axis and turns around ONLY at $x = -2$.
c. y-INTERCEPT. $f(0) = 0$, so the y intercept is $y = 0$.
d. SYMMETRY. Since $f(-x) = (-x)^3((-x)+2)^2((-x)+1) = -x^3(x-2)^2(1-x)$ which is not equal to $f(x)$ or to $-f(x)$. Therefore the graph has no symmetry.
e. The graph looks like



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7. For the following polynomial functions, find

1. The end behaviour.
2. The x-intercepts and their multiplicity.
3. The y-intercept.
4. Whether the function is even, odd or neither.

a) $f(x) = x^3 - 13x + 12$

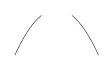
b) $f(x) = -2x^4 + 2x^2$

c) $f(x) = x^7 - 3x^6 - x^5 + 11x^4 - 12x^3 + 4x^2$

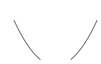
d) $x^4 - 2x^3 - x^2 + 4x - 2$

Solutions:

a)
1. End behaviour: The leading term is x^3, so for x large $f(x)$ looks like x^3: 
2. x-intercepts: $x^3 - 13x + 12 = (x-3)(x-1)(x+4)$, so the x-intercepts are 3, 1 and -4, all with multiplicity 1.
3. The y-intercept: $f(0) = 12$.
4. Is neither. If it were, the x-intercepts would be symmetric about the y-axis.

b)
1. End behaviour: The leading term is $-2x^4$, so for x large $f(x)$ looks like $-2x^4$: 
2. x-intercepts: $-2x^4 + 2x^2 = -2x^2(x-1)(x+1)$, so the x-intercepts are
   - 0 with multiplicity 2.
   - 1 and -1 with multiplicity 1.
3. The y-intercept: $f(0) = 0$.
4. $f(-x) = -2(-x)^4 + 2(-x)^2 = -2x^4 + 2x^2 = f(x)$, so f is even (in other words, symmetric about the y-axis).

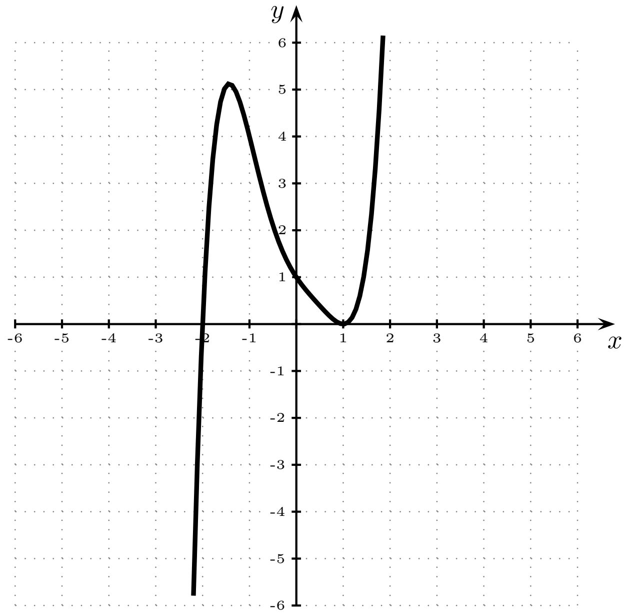
c)
1. End behaviour: The leading term is x^7, so for x large $f(x)$ looks like x^7: 
2. x-intercepts: $x^7 - 3x^6 - x^5 + 11x^4 - 12x^3 + 4x^2 = x^2(x-2)(x-1)^3(x+2)$ (use synthetic division), so the x-intercepts are
   - 0 with multiplicity 2.
   - 1 with multiplicity 3.
   - 2 and -2 with multiplicity 1.
3. The y-intercept: $f(0) = 0$.
4. Is neither. If it were, the x-intercepts would be symmetric about the y-axis.

d)
1. End behaviour: The leading term is x^4, so for x large $f(x)$ looks like x^4: 
2. x-intercepts: $x^4 - 2x^3 - x^2 + 4x - 2 = (x-1)^2(x^2-2)$ (use synthetic division), so the x-intercepts are
   - 1 with multiplicity 2.
   - $\sqrt{2}$ and $-\sqrt{2}$ with multiplicity 1 (these are the solutions of $(x^2-2) = 0$).
3. The y-intercept: $f(0) = -2$.

4. Is neither. If it were, the x-intercepts would be symmetric about the y-axis (and 1 is an x-intercept, but -1 is not).

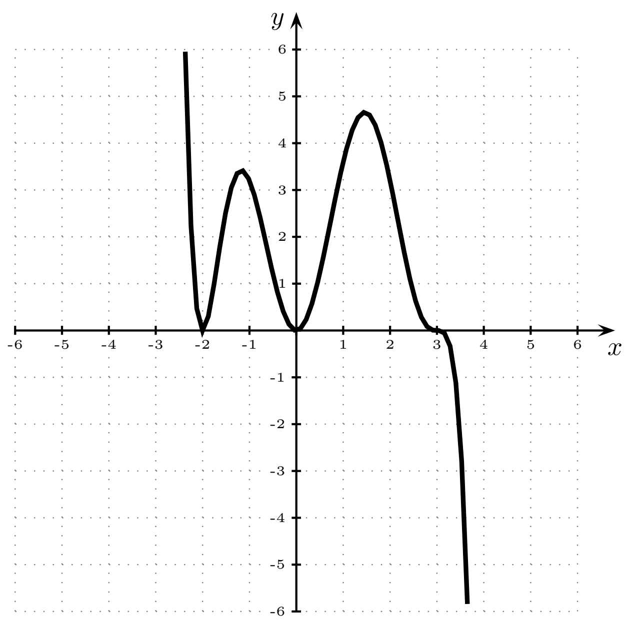
8. Given the following information about the polynomial function $f(x)$, graph $f(x)$ in the axes provided.

1. End behaviour: The leading term of $f(x)$ is x^5.
2. x-intercepts:
   - 1 with multiplicity 2.
   - -2 with multiplicity 1.
3. y-intercept: $f(0) = 1$.
4. Symmetry: f is not even nor odd.



9. Given the following information about the polynomial function $f(x)$, graph $f(x)$ in the axes provided.

1. End behaviour: The leading term of $f(x)$ is $-x^7$.
2. x-intercepts:
   - -2 with multiplicity 2.
   - 0 with multiplicity 2.
   - 3 with multiplicity 3.
3. y-intercept: $f(0) = 0$.
4. Symmetry: f is not even nor odd.



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10. For the following rational functions, first find

1. The end behaviour and the horizontal asymptotes, if any.
2. The vertical asymptotes.
3. The x-intercepts and their multiplicity.
4. The y-intercept.

Then **sketch the graph of the function** in the graph paper provided (or in your own).

a) $f(x) = \dfrac{x+1}{x-1}$

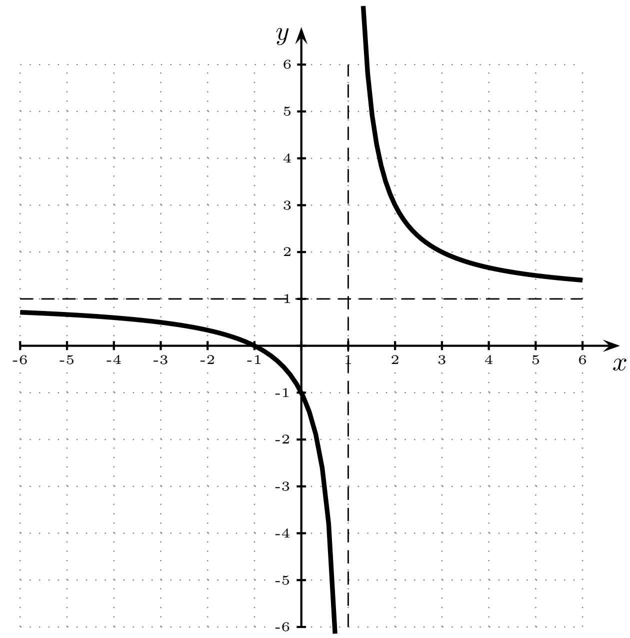
b) $f(x) = \dfrac{3x^2}{x^2-9}$

c) $f(x) = \dfrac{x-4}{x^2-x-6}$

$f(x) = \dfrac{2x+5}{x^3-13x+12}$

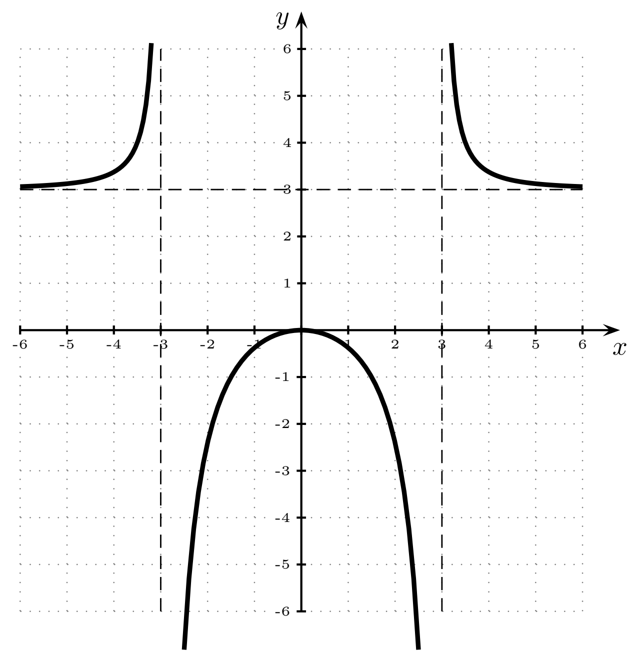
Solutions:

a)



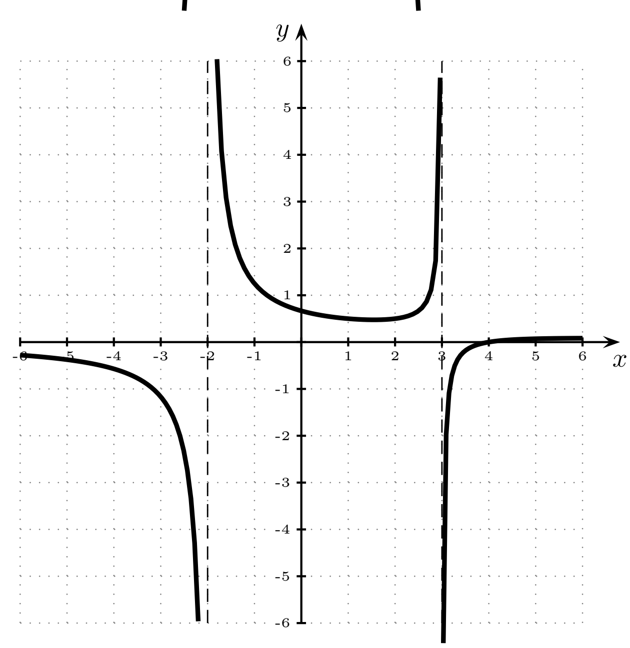
1. End behaviour: As $x \to \pm\infty$, $f(x) \approx \frac{x}{x} = 1$, so it has a horizontal asymptote at $y = 1$.
2. Vertical asymptotes: the denominator is $x - 1$; it is 0 only when $x = 1$, so f has a vertical asymptote at $x = 1$.
3. x-intercepts: the numerator is $x + 1$; it is 0 only when $x = -1$, so the only x-intercept is $x = -1$ with multiplicity 1.
4. y-intercept: $f(0) = \dfrac{0+1}{0-1} = -1$.

b)



1. End behaviour: As $x \to \pm\infty$, $f(x) \approx \dfrac{3x^2}{x^2} = 3$, so it has a horizontal asymptote at $y = 3$.
2. Vertical asymptotes: the denominator is $x^2 - 9$; it is 0 only when $x = 3$ or $x = -3$, so f has vertical asymptotes at $x = 3$ and at $x = -3$.
3. x-intercepts: the numerator is x^2; it is 0 only when $x = 0$, so the only x-intercept is $x = 0$ with multiplicity 2.
4. y-intercept: $f(0) = 0$.

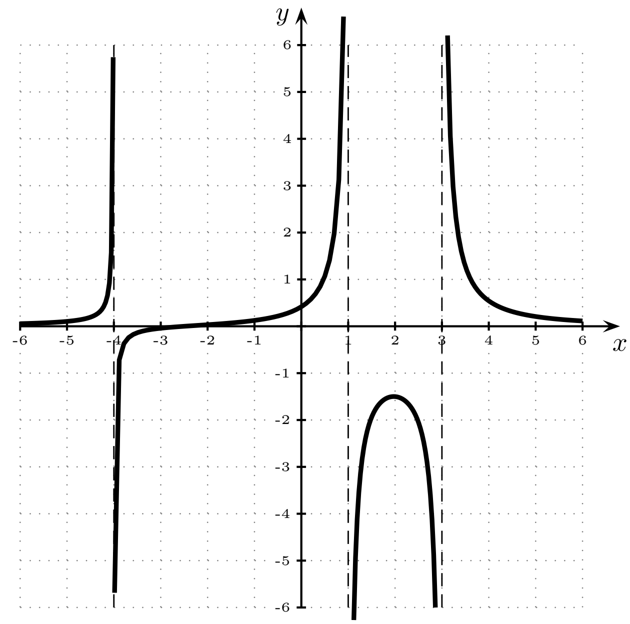
c)



1. End behaviour: As $x \to \pm\infty$, $f(x) \approx \dfrac{x}{x^2} = \dfrac{1}{x}$, which has a horizontal asymptote at $y = 0$. Therefore f has also a horizontal asymptote at $y = 0$.
2. Vertical asymptotes: the denominator is $x^2 - x - 6 = (x-3)(x+2)$, so f has vertical asymptotes at $x = 3$ and $x = -2$.
3. x-intercepts: the numerator is $x - 4$, so the only x-intercept is $x = 4$ with multiplicity 1.
4. y-intercept: $f(0) = \dfrac{0-4}{0^2 - 0 - 6} = \dfrac{-4}{-6} = \dfrac{2}{3}$.

d)

1. <u>End behaviour</u>: As $x \to \pm\infty$, $f(x) \approx \frac{2x}{x^3} = \frac{2}{x^2}$, which has a horizontal asymptote at $y = 0$. Therefore f has also a horizontal asymptote at $y = 0$.
2. <u>Vertical asymptotes</u>: the denominator is $x^3 - 13x + 12 = (x-1)(x-3)(x+4)$, so f has vertical asymptotes at $x = 3$ and $x = -4$ and $x = 1$.
3. <u>x-intercepts</u>: the numerator is $2x+5$, so the only x-intercept is $x = -\frac{5}{2} = -2.5$ with multiplicity 1.
4. <u>y-intercept</u>: $f(0) = \frac{5}{12}$.



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11. Given the following information about the rational function $f(x)$, graph $f(x)$ in the axes provided.

1. <u>End behaviour</u>: Horizontal asymptote at $y = 1$.
2. <u>End behaviour</u>: Vertical asymptotes:
   - At $x = -3$.
   - $x = 3$.
3. <u>x-intercepts</u>:
   - -2 with multiplicity 1.
   - 2 with multiplicity 1.
4. <u>y-intercept</u>: $f(0) = 1$.
5. <u>Symmetry</u>: f is an even function.