9.3. Quadratic equations and the quadratic formula Professor Luis Fernández

Quadratic equation.

Recall: A quadratic equation is a polynomial equation of degree 2 (that is, the highest power of the variable that appears in the equation is 2).

We have seen many quadratic equations, and we have learned a method to solve them, namely factoring. Let us recall the method:

1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.
2. Factor the quadratic expression on the left.
3. Use the Zero Product Property (that is, set each factor equal to 0).
4. Solve the two linear equations you obtained in 3.
5. Check. Substitute each solution separately into the original equation.

Unfortunately, this method works well with some simple equations, but it cannot be used for most of them because most polynomials cannot be factored with the methods that we have studied.

Now we will study a new method. More than a method, it is just a formula that gives the solutions at once! You only have to evaluate the formula and you are done. The great thing about this method is that it works for *every* quadratic equation.

The quadratic formula

The two solutions of the equation $ax^2 + bx + c = 0$ are given by the formulas

$$\boxed{x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.}$$

Let us use this formulas in an example.

Example: Solve the equation $x^2 + 20x + 91 = 0$.

In this equation, $a = 1$, $b = 20$ and $c = 91$. Therefore the solutions are

$$x = \frac{-20 + \sqrt{20^2 - 4 \cdot 1 \cdot 91}}{2 \cdot 1} \qquad \text{and} \qquad x = \frac{-20 - \sqrt{20^2 - 4 \cdot 1 \cdot 91}}{2 \cdot 1}.$$

It only remains to simplify the expressions above.

Let us start simplifying the radicand: $20^2 - 4 \cdot 1 \cdot 91 = 400 - 364 = 36$.

Therefore, one of the solutions is

$$x = \frac{-20 + \sqrt{36}}{2} = \frac{-20 + 6}{2} = \frac{-14}{2} = -7,$$

and the other one is

$$x = \frac{-20 - \sqrt{36}}{2} = \frac{-20 - 6}{2} = \frac{-26}{2} = -13.$$

That's it!

It is simpler to do it in steps, as follows:

1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$ and identify a, b and c.
2. Find the *discriminant*: $D = b^2 - 4ac$.
3. Simplify $\sqrt{D}$.
4. Substitute a, b, and $\sqrt{D}$ (found in step 3) into the formulas $x = \dfrac{-b + \sqrt{D}}{2a}$ and $x = \dfrac{-b - \sqrt{D}}{2a}$ and simplify.

Example: Solve the equation $3x^2 + 4x - 5 = 0$.

1. $a = 3$, $b = 4$ and $c = -5$.

2. The discriminant is $D = 4^2 - 4 \cdot 3 \cdot (-5) = 16 + 60 = 76$.

3. Simplify $\sqrt{D} = \sqrt{76} = \sqrt{4}\sqrt{19} = 2\sqrt{19}$.

4. One solution is $x = \dfrac{-4 + 2\sqrt{19}}{2 \cdot 3} = \dfrac{2(-2 + \sqrt{19})}{6} = \dfrac{\cancel{2}(-2 + \sqrt{19})}{\cancelto{3}{6}} = \dfrac{-2 + \sqrt{19}}{3}$.

The other one is $x = \dfrac{-2 - \sqrt{19}}{3}$ (you just need to change the sign of the radical!).

NOTE: You need to memorize the quadratic formula.

Since the only difference between the two formulas is a "$-$" instead of a "$+$", one normally writes the quadratic formula more concisely as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

That is, one solution is obtained with the "$+$" and another with the "$-$".

Exercise: Write down **10 times** on your notebook the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Some quadratic equations do not have solutions that are real numbers: they are complex numbers. This is one of the main reasons for the invention of the complex numbers: to guarantee that every quadratic equation has a solution.

Example: Solve the equation $x^2 + x = -1$.

1. Write it as $ax^2 + bx + c =$: Just add 1 to both sides to get $x^2 + x + 1 = 0$. Thus, $a = 1$, $b = 1$ and $c = 1$.

2. The discriminant is $D = 1^2 - 4 \cdot 1 \cdot 1 = 1 - 4 = -3$.

3. Simplify $\sqrt{D} = \sqrt{-3} = \sqrt{3}\sqrt{-1} = i\sqrt{3}$.

4. One solution is $x = \dfrac{1 + i\sqrt{3}}{2 \cdot 1} = \dfrac{1 + i\sqrt{3}}{2}$.

The other one is $x = \dfrac{1 - i\sqrt{3}}{2}$.

Exercises: Solve the following quadratic equations, making sure to simplify the solutions completely.

5. $4x^2 + x - 3 = 0$. **6.** $3x^2 + 8x - 3 = 0$. **7.** $x^2 + 2x + 6 = 0$.

8. $4x^2 + 4x + 1 = 0$. **9.** $x^2 - 8x = 33$. **10.** $2x^2 + 1 = x$.

11. $(x + 1)(x - 3) = 2$. **12.** $(x + 4)(x - 7) = 18$. **13.** $2x^2 + 3x + 3 = 0$.

Some particular cases

When $a = 0$, the equation is not quadratic (it is linear), since x^2 does not appear in the equation.

When $b = 0$, or in general if x^2 appears in the equation but x does not, then it is easier to solve the equation by isolating x^2 (say on the left hand side) and then taking square roots of both sides. The two solutions will be $+$ the square root of the right hand side **and** $-$ the square root of the right hand side.

Example: Solve $3x^2 - 5 = 0$.

Since x does not appear in the equation, let us isolate x^2 on the left hand side:

Add 5 to both sides and then divide by 3 to get $x^2 = \dfrac{5}{3}$.

Then we only have to take square roots of both sides, to get $x = \pm\dfrac{\sqrt{5}}{\sqrt{3}} = \pm\dfrac{\sqrt{15}}{3}$.

When $c = 0$, both terms of the equation have a factor of x, so x can be factored out and the equation can be easily solved by factoring, as in chapter 6.

Exercises: Solve the following quadratic equations, making sure to simplify the solutions completely.

14. $x^2 - 3 = 0$.

15. $3x^2 = 15$.

16. $5x^2 + 2x = 0$.

17. $4x^2 = 3x$.

18. $x^2 + 5 = 0$.

19. $2x^2 + 1 = x^2$

20. $(x + 1)(x - 3) = 2$.

21. $(x + 4)(x - 7) = 18$.

22. $2x^2 + 3x + 3 = 0$.