8.4. Add, subtract, and multiply rational expressions. Professor Luis Fernández

Addition and subtraction of rational expressions.

In general, addition or subtraction and radicals "do not mix well", meaning that there is not a kind of distributive rule between these operations (the radical of the sum is **NOT** the sum of the radicals).

This means that sums inside a radical **cannot be distributed**; they stay as they are.

It also means that when we add radicals with different index or different radicand there is no way to simplify them.

However, when the two radicals have the same index and the same radicand they can be added or subtracted, essentially the same way as we combined like terms with polyomials. For example, $3\sqrt{5}+4\sqrt{5}=7\sqrt{5}$. Same as we had "like terms" when we talked about polynomials, here we have "like radicals":

Like radicals are radical expressions with the same index and the same radicand.

For example, $3\sqrt{5}$ and $2\sqrt{5}$ are like radicals. $6\sqrt[4]{7}$ and $-46\sqrt[4]{7}$ are like radicals. $-3\sqrt[3]{2}$ and $2\sqrt[3]{2}$ are like radicals.

To add or subtract radical expressions we combine all like radicals as we did when we combined like terms.

Example: Add: $3\sqrt{5}+9\sqrt{5}$.

Since the two terms are like radicals, we only need to combine them: $3\sqrt{5}+9\sqrt{5}=12\sqrt{5}$.

Example: Simplify $\sqrt{3}-4\sqrt{6}+2\sqrt{3}+3\sqrt{6}$.

$\sqrt{3}$ and $2\sqrt{3}$ are like radicals, so we can combine them to get $3\sqrt{3}$. On the other hand, $-4\sqrt{6}$ and $3\sqrt{6}$ are like radicals, so we can combine them to get $-\sqrt{6}$. Therefore,

$$\sqrt{3}-4\sqrt{6}+2\sqrt{3}+3\sqrt{6}=3\sqrt{3}-\sqrt{6}.$$

Notice that the two terms of the last expression cannot be combined because they are not like radicals.

Sometimes radicals that do not seem to be like radicals at first, but they are after simplifying them as we learned in the previous section:

Example: Add $\sqrt{27}-4\sqrt{3}$.

Simplify and notice that the two terms are like radicals; then combine them:
$\sqrt{27}-4\sqrt{3}=\sqrt{9\cdot 3}-4\sqrt{3}=3\sqrt{3}-4\sqrt{3}=-\sqrt{3}$.

Example: Simplify $7\sqrt[3]{5}-2\sqrt[3]{40}$.

Notice that $\sqrt[3]{40}=\sqrt[3]{8\cdot 5}=\sqrt[3]{8}\cdot\sqrt[3]{5}=2\sqrt[3]{5}$. Therefore $7\sqrt[3]{5}-2\sqrt[3]{40}=7\sqrt[3]{5}-2\cdot 2\cdot\sqrt[3]{5}=7\sqrt[3]{5}-4\sqrt[3]{5}=3\sqrt[3]{5}$.

When there are variables, it is exactly the same: simplify first and then combine like radicals:

Example: Simplify $\sqrt[3]{135x^7}-\sqrt[3]{40x^7}-\sqrt[3]{5x^2}$.

Notice that $135=27\cdot 5$ and $x^7=x^6\cdot x$, so we can simplify $\sqrt[3]{135x^7}=\sqrt[3]{27x^6\cdot 5x}=\sqrt[3]{27x^6}\cdot\sqrt[3]{5x}=3x^2\sqrt[3]{5x}$.
Likewise, $\sqrt[3]{40x^7}=2x^2\sqrt[3]{5x}$.
The last radical, $\sqrt[3]{5x^2}$, is already simplified.
Therefore $\sqrt[3]{135x^7}-\sqrt[3]{40x^7}-\sqrt[3]{5x}=3x^2\sqrt[3]{5x}-2x^2\sqrt[3]{5x}-\sqrt[3]{5x}=x^2\sqrt[3]{5x}-\sqrt[3]{5x^2}$.
Notice that the last radical, that is $\sqrt[3]{5x^2}$, is not like the others, so it cannot be combined.

Exercises: Simplify the following expressions.

1. $5\sqrt{2}-3\sqrt{2}$ **2.** $6\sqrt[3]{6}+7\sqrt[3]{6}$ **3.** $-12\sqrt[5]{18}+7\sqrt[5]{18}$ **4.** $4\sqrt[3]{5}-6\sqrt{5}$

5. $3\sqrt{2x}+5\sqrt{2x}$ **6.** $-5\sqrt[3]{5x^2}+7\sqrt[3]{5x^2}$ **7.** $-4\sqrt[4]{7xy^2}+5\sqrt[4]{7xy^2}$ **8.** $4\sqrt[3]{10x^2}-6\sqrt[3]{10x}$

9. $\sqrt{27}+\sqrt{75}$ **10.** $\sqrt[3]{24}-\sqrt[3]{81}$ **11.** $\sqrt{72}+\sqrt{98}$ **12.** $4\sqrt{45}-\sqrt{80}$

13. $\sqrt{72x^5}-\sqrt{50x^5}$ **14.** $\sqrt[3]{64x^6}-3\sqrt[3]{125x^6}$ **15.** $\sqrt{48x^5}+\sqrt{75x^5}$ **16.** $4\sqrt[4]{243y^6}+2\sqrt[4]{3y^6}$

Multiplication of combinations of radical expressions

As with addition and subtraction, multiplying combinations of radicals is essentially the same as multiplying polynomials: use the distibutive law, multiply and then simplify. For example:

$$(2+\sqrt{3})(5-\sqrt{3}) = 2\cdot 5 - 2\cdot\sqrt{3} + 5\cdot\sqrt{3} - \sqrt{3}\cdot\sqrt{3} = 10 + 3\sqrt{3} - \sqrt{9} = 7 + 3\sqrt{3}.$$

Example: Multiply $\sqrt{6}(\sqrt{2}+\sqrt{18})$.

$\sqrt{6}(\sqrt{2}+\sqrt{18}) = \sqrt{6}\cdot\sqrt{2}+\sqrt{6}\cdot\sqrt{18} = \sqrt{4\cdot 3}+\sqrt{36\cdot 3} = 2\sqrt{3}+6\sqrt{3} = 8\sqrt{3}$.

Example: Multiply $\sqrt[3]{3}(-\sqrt[3]{9}-\sqrt[3]{6})$.

$\sqrt[3]{3}(-\sqrt[3]{9}-\sqrt[3]{6}) = -\sqrt[3]{3}\cdot\sqrt[3]{9}-\sqrt[3]{3}\cdot\sqrt[3]{6} = -\sqrt[3]{27}-\sqrt[3]{18} = -3-\sqrt[3]{18}$.

Example: Multiply $(\sqrt{6}-3\sqrt{8})(2\sqrt{6}+\sqrt{8})$.

$$\begin{aligned}(\sqrt{6}-3\sqrt{8})(2\sqrt{6}+\sqrt{8}) &= 2\sqrt{6}\cdot\sqrt{6}+\sqrt{6}\cdot\sqrt{8}-6\sqrt{8}\cdot\sqrt{6}-3\sqrt{8}\cdot\sqrt{8}\\ &= 2\sqrt{36}+\sqrt{16\cdot 3}-6\sqrt{16\cdot 3}-3\sqrt{64}\\ &= 2\cdot 6+4\sqrt{3}-6\cdot 4\sqrt{3}-3\cdot 8\\ &= 12-24+4\sqrt{3}-24\sqrt{3} = -12-20\sqrt{3}.\end{aligned}$$

There are some special products that we studied before. Recall:

- $(a+b)^2 = a^2+2ab+b^2$.
- $(a-b)^2 = a^2-2ab+b^2$.
- $(a+b)(a-b) = a^2-b^2$.

This helps when doing operations with radicals:

Example: Simplify $(3+2\sqrt{5})^2$.

$(3+2\sqrt{5})^2 = 3^2+2\cdot 2\sqrt{5}+(2\sqrt{5})^2 = 9+4\sqrt{5}+4\sqrt{25} = 9+4\sqrt{5}+20 = 29+4\sqrt{5}$.

Example: Simplify $(3+\sqrt{5})(3-\sqrt{5})$.

$(3+\sqrt{5})(3-\sqrt{5}) = 3^2-(\sqrt{5})^2 = 9-5 = 4$.

Notice that, while the original factors have roots, the final product is a whole number. This fact will be used in the next section.

Exercises: Simplify the following expressions.

17. $\sqrt{7}(5+2\sqrt{7})$ **18.** $\sqrt[3]{6}(4+2\sqrt[3]{18})$ **19.** $\sqrt[4]{2}(\sqrt[4]{12}+\sqrt[4]{24})$

20. $\sqrt{2}(-5+9\sqrt{2}$ **21.** $(7+\sqrt{3})(9-\sqrt{3})$ **22.** $(7+\sqrt[3]{6})(4+2\sqrt[3]{18})$

23. $(\sqrt{3}+\sqrt{10})(\sqrt{3}+2\sqrt{10})$ **24.** $(2\sqrt{7}-5\sqrt{11})(4\sqrt{7}+9\sqrt{11})$ **25.** $(\sqrt{11}+\sqrt{5})(\sqrt{11}+6\sqrt{5})$

26. $(\sqrt{3}+\sqrt{5})^2$ **27.** $(\sqrt{2}-7)^2$ **28.** $(\sqrt{6}+\sqrt{2})^2$

29. $(\sqrt{3}+\sqrt{5})(\sqrt{3}-\sqrt{5})$ **30.** $(\sqrt{2}-\sqrt{7})(\sqrt{2}+\sqrt{7})$ **31.** $(2+\sqrt{5})(2-\sqrt{5})$

32. $(2\sqrt{6}+3\sqrt{2})(2\sqrt{6}-3\sqrt{2})$ **33.** $(3\sqrt{3}-2\sqrt{2})(3\sqrt{2}+2\sqrt{2})$ **34.** $(2+3\sqrt{7})(2-3\sqrt{7})$