8.3. Rational exponents. Multiplying simple radicals. Professor Luis Fernández

Rational exponents.

Recall: if n is a positive integer, $a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ times}}$, and $a^{-n} = \dfrac{1}{a^n}$.

Therefore, we know what expressions like 3^4 or 5^{-2} mean.

What about, for example, $4^{\frac{3}{2}}$? What is the meaning of having a fraction as exponent? Here is the definition:

$$\boxed{a^{\frac{p}{q}} \quad \text{means} \quad \sqrt[q]{a^p} \text{ or } (\sqrt[q]{a})^p}$$

Of course, the two definitions are equal and you can choose the most convenient in each case. That is,

$$\sqrt[q]{a^p} = (\sqrt[q]{a})^p.$$

(Rigurously, if q is even and p is odd, one has to require the number a to be positive, but do not worry about it.)

For example, $4^{\frac{3}{2}} = (\sqrt[2]{4})^3 = 2^3 = 8$.

For example, $5^{\frac{4}{3}} = \sqrt[3]{5^4} = (\sqrt[3]{5})^4$.

If the exponent is also negative, one only has to do the reciprocal:

For example, $6^{-\frac{3}{5}} = \dfrac{1}{\sqrt[5]{6^3}} = \dfrac{1}{(\sqrt[5]{6})^3}$

Let us practice rewriting expressions with rational exponents.

Exercises: Rewrite (in both ways) using radicals and integer exponents:

1. $x^{\frac{2}{3}}$	**2.** $x^{\frac{6}{7}}$	**3.** $x^{\frac{9}{2}}$	**4.** $x^{\frac{2}{5}}$
5. $x^{\frac{1}{3}}$	**6.** $x^{\frac{1}{2}}$	**7.** $x^{\frac{1}{4}}$	**8.** $x^{\frac{1}{5}}$
9. $x^{-\frac{1}{3}}$	**10.** $x^{-\frac{1}{2}}$	**11.** $x^{-\frac{1}{4}}$	**12.** $x^{-\frac{1}{5}}$
13. $x^{-\frac{7}{3}}$	**14.** $x^{-\frac{4}{5}}$	**15.** $x^{-\frac{3}{2}}$	**16.** $x^{\frac{2}{3}}$

Exercises: Rewrite using rational exponents:

17. $\sqrt{x^3}$	**18.** $\sqrt{x^5}$	**19.** $\sqrt[3]{x^5}$	**20.** $\sqrt[5]{x^3}$
21. $(\sqrt{x})$	**22.** $\sqrt[3]{x}$	**23.** $\sqrt[5]{x}$	**24.** $\sqrt[7]{x}$
25. $(\sqrt{x})^3$	**26.** $(\sqrt[3]{x})^5$	**27.** $(\sqrt[3]{x})^{-5}$	**28.** $\sqrt[5]{x^{-3}}$
29. $\left(\dfrac{1}{\sqrt{x}}\right)^3$	**30.** $\dfrac{1}{(\sqrt[6]{x})^5}$	**31.** $\dfrac{1}{(\sqrt[3]{x})^4}$	**32.** $\left(\dfrac{1}{\sqrt[5]{x}}\right)^{-2}$

There are several great things about rational exponents. First, it broadens the definition of exponents to include all rational numbers. Second, and more important for us, makes multiplication of radicals much easier. Further, the rules of exponents work the same. Recall the rules of exponents:

1. Product Property	$a^m \cdot a^n = a^{m+n}$.	5. Quotient to a Power Property	$\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$
2. Quotient Property	$\dfrac{a^m}{a^n} = a^{m-n}$	6. Zero Exponent Definition	$a^0 = 1$
3. Power Property	$(a^m)^n = a^{m \cdot n}$	7. Definition of Negative Exponent	$a^{-n} = \dfrac{1}{a^n}$
4. Product to a Power	$(ab)^m = a^m b^m$	8. Property of Negative Exponents	$\left(\dfrac{a}{b}\right)^{-m} = \left(\dfrac{b}{a}\right)^m$

For example, $x^{\frac{2}{3}} \cdot x^{\frac{3}{4}} = x^{\frac{2}{3}+\frac{3}{4}}$. Now, $\dfrac{2}{3} + \dfrac{3}{4} = \dfrac{8}{12} + \dfrac{9}{12} = \dfrac{17}{12}$. Therefore $x^{\frac{2}{3}} \cdot x^{\frac{3}{4}} = x^{\frac{17}{12}}$.

Thus, to multiply or divide radical expressions, one can convert the expressions into rational exponents and multiply or divide using the rules of exponents, which amounts to just adding or subtracting the exponents. At the end, convert back to radicals and simplify.

Example: Multiply: $\sqrt[4]{x^3} \cdot \sqrt[6]{x^5}$.

$$\sqrt[4]{x^3} \cdot \sqrt[6]{x^5} = x^{\frac{3}{4}} \cdot x^{\frac{5}{6}} = x^{\frac{3}{4}+\frac{5}{6}}. \quad \text{Now, } \frac{3}{4} + \frac{5}{6} = \frac{9}{12} + \frac{10}{12} = \frac{19}{12}.$$

Therefore, $\sqrt[4]{x^3} \cdot \sqrt[6]{x^5} = x^{\frac{19}{12}} = \sqrt[12]{x^{19}}$, and simplifying, we get $\sqrt[12]{x^{19}} = \sqrt[12]{x^{12} \cdot x^7} = x \cdot \sqrt[12]{x^7}$.

Example: Divide: $\dfrac{\sqrt[3]{x^4}}{\sqrt{x^3}}$.

$$\frac{\sqrt[3]{x^4}}{\sqrt{x^3}} = \frac{x^{\frac{4}{3}}}{x^{\frac{3}{2}}} = x^{\frac{4}{3}-\frac{3}{2}}. \quad \text{Now, } \frac{4}{3} - \frac{3}{2} = \frac{8}{6} - \frac{9}{6} = -\frac{1}{6}, \text{ which implies that } x^{\frac{4}{3}-\frac{3}{2}} = x^{-\frac{1}{6}}.$$

Therefore, $\dfrac{\sqrt[3]{x^5}}{\sqrt{x^4}} = x^{-\frac{1}{6}} = \dfrac{1}{\sqrt[6]{x}}$.

Exercises: Multiply and simplify. Write the answer using radicals.

33. $x^{\frac{2}{3}} \cdot x^{\frac{7}{3}}$ **34.** $x^{\frac{6}{7}} \cdot x^{\frac{2}{7}}$ **35.** $x^{\frac{3}{7}}$ **36.** $x^{\frac{2}{5}} \cdot x^{\frac{2}{5}}$

37. $x^{\frac{1}{6}} \cdot x^{\frac{1}{4}}$ **38.** $x^{\frac{1}{2}} \cdot x^{-\frac{3}{5}}$ **39.** $x^{-\frac{5}{6}} \cdot x^{-\frac{3}{8}}$ **40.** $x^{-\frac{1}{5}} \cdot x^{\frac{1}{5}}$

41. $\sqrt{x^3} \cdot \sqrt{x}$ **42.** $\sqrt[3]{x} \cdot \sqrt[3]{x^5}$ **43.** $\sqrt[4]{x^3} \cdot \sqrt[6]{x^5}$ **44.** $\sqrt[5]{x^3} \cdot \sqrt[3]{x^2}$

Multiplication of simple radicals

Although we have seen how to multiply expressions by converting them into rational exponents, when the index of the two factors is the same it is easier to do it directly using the multiplication rule for radicals:

$$\boxed{\sqrt[n]{x} \cdot \sqrt[n]{y} = \sqrt[n]{x \cdot y}.}$$

Of course **this rule only works when the index of the two factors is the same**. If it is not, one has to convert them into rational exponents and proceed as above.

Example: Multiply $\sqrt[3]{x^5} \cdot \sqrt[3]{x^4}$.

$$\sqrt[3]{x^5} \cdot \sqrt[3]{x^4} = \sqrt[3]{x^5 \cdot x^4} = \sqrt[3]{x^9} = x^3.$$

Example: Multiply $\sqrt{x^3} \cdot \sqrt{x^4}$.

$$\sqrt{x^3} \cdot \sqrt{x^4} = \sqrt{x^3 \cdot x^4} = \sqrt{x^7} = \sqrt{x^6 \cdot x} = x^3 \cdot \sqrt{x}.$$

When there are several variables, or variables with numbers, just multiply each part and simplify:

Example: Multiply $\sqrt{8x^3y^2} \cdot \sqrt{6xy^5}$.

$$\sqrt{8x^3y^2} \cdot \sqrt{6xy^5} = \sqrt{8x^3y^2 \cdot 6xy^5} = \sqrt{16 \cdot 3 \cdot x^4 \cdot y^7} = 4x^2\sqrt{3 \cdot y^6 \cdot y} = 4x^2y^3\sqrt{3y}$$

Exercises: Multiply and simplify.

45. $\sqrt{x^3} \cdot \sqrt{x}$ **46.** $\sqrt[3]{x} \cdot \sqrt[3]{x^5}$ **47.** $\sqrt[4]{x^3} \cdot \sqrt[4]{x^3}$ **48.** $\sqrt[5]{x^3} \cdot \sqrt[5]{x^6}$

49. $\sqrt{6x^2} \cdot \sqrt{2x}$ **50.** $\sqrt[3]{3x^4y} \cdot \sqrt[3]{10x^5y^2}$ **51.** $\sqrt[4]{20x^3y^5} \cdot \sqrt[4]{12x^3y}$ **52.** $\sqrt[3]{4x^2y} \cdot \sqrt[3]{12x^5y^3}$

53. $\sqrt{6x^3y} \cdot \sqrt{2x^2y}$ **54.** $\sqrt[3]{25xy^2} \cdot \sqrt[3]{10x^5y}$ **55.** $\sqrt[4]{9x^2y} \cdot \sqrt[4]{18x^3y^2}$ **56.** $\sqrt[3]{49x^2y^2} \cdot \sqrt[3]{7xy^3}$