7.3. Simplifying complex rational expressions Professor Luis Fernández

Review: Fractions of fractions

Recall: An expression of the form $\frac{\frac{a}{b}}{\frac{c}{d}}$ means exactly the same as $\frac{a}{b} \div \frac{c}{d}$. However, it is common to write it using the first way. In this sections we will work with rational expressions that themselves have rational expressions as numerator and denominator. For example

$$\frac{\frac{1}{x} + \frac{x+1}{x+2}}{\frac{x-1}{x} - \frac{1}{x+2}}.$$

Let us first do some exercises to review how it is done with fractions of numbers.

Example: Simplify $\dfrac{\frac{2}{3} + \frac{3}{4}}{2 - \frac{5}{8}}$.

First simplify the numerator of the big fraction:

$$\frac{2}{3} + \frac{3}{4} = \frac{8}{12} + \frac{9}{12} = \frac{17}{12}.$$

Then simplify the denominator:

$$2 - \frac{5}{8} = \frac{2}{1} - \frac{5}{8} = \frac{16}{8} - \frac{5}{8} = \frac{11}{8}.$$

Then divide:

$$\frac{\frac{2}{3} + \frac{3}{4}}{2 - \frac{5}{8}} = \frac{\frac{17}{12}}{\frac{11}{8}} = \frac{17}{12} \div \frac{11}{8} = \frac{17}{12} \cdot \frac{8}{11} = \frac{17 \cdot \overset{2}{\cancel{8}}}{\underset{3}{\cancel{12}} \cdot 11} = \frac{17 \cdot 2}{3 \cdot 11} = \frac{34}{33}.$$

Therefore, one way to simplify these expressions is to:

- Simplify the numerator and the denominator of the big fraction separately so that each is written as a single fraction.
- Convert the big fraction into multiplication of the reciprocal of the denominator.

Practice exercises: Simplify the following by first simplifying the numerator and the denominator

1. $\dfrac{\frac{7}{18}}{\frac{35}{12}}$ **2.** $\dfrac{\frac{3}{4} + \frac{1}{6}}{\frac{3}{8} - \frac{1}{6}}$ **3.** $\dfrac{\frac{1}{4} - \frac{1}{3}}{\frac{2}{15} + \frac{3}{10}}$ **4.** $\dfrac{\frac{3}{4} + \frac{1}{6}}{\frac{3}{8} - \frac{1}{6}}$

There is another, perhaps easier, way to do the exercise above. The idea is to multiply the numerator and denominator of the big fraction by the LCM of all the denominators. Let us do the previous exercise using this method:

Example (version 2): Simplify $\dfrac{\frac{2}{3} + \frac{3}{4}}{2 - \frac{5}{8}}$.

The denominators are 1, 3, 4, and 8, and the LCM of these numbers is 24. Let us multiply numerator and denominator of the big fraction by 24 and simplify:

$$\frac{\frac{2}{3} + \frac{3}{4}}{2 - \frac{5}{8}} = \frac{\frac{24}{1}\left(\frac{2}{3} + \frac{3}{4}\right)}{\frac{24}{1}\left(2 - \frac{5}{8}\right)} = \frac{\frac{2 \cdot 24}{3} + \frac{3 \cdot 24}{4}}{2 \cdot 24 - \frac{5 \cdot 24}{8}} = \frac{\frac{2 \cdot \overset{8}{\cancel{24}}}{\underset{1}{\cancel{3}}} + \frac{3 \cdot \overset{6}{\cancel{24}}}{\underset{1}{\cancel{4}}}}{2 \cdot 24 - \frac{5 \cdot \overset{3}{\cancel{24}}}{\underset{1}{\cancel{8}}}} = \frac{2 \cdot 8 + 3 \cdot 6}{2 \cdot 24 - 5 \cdot 3} = \frac{34}{33}.$$

The advantage is that all the denominators cancel out so you do not have to add or subtract any fractions! Summarizing, to simplify a complex rational expression using the LCD:

- Find the LCD of all the denominators in the big fraction.
- Multiply the numerator and denominator of the big fraction by the LCD and simplify. All the denominators will dissappear and you will end up with a single fraction.
- Simplify the fraction if necessary.

Practice exercises: Simplify the following by multiplying by the LCD.

5. $\dfrac{\frac{7}{18}}{\frac{35}{12}}$ **6.** $\dfrac{\frac{3}{4}+\frac{1}{6}}{\frac{3}{8}-\frac{1}{6}}$ **7.** $\dfrac{\frac{1}{4}-\frac{1}{3}}{\frac{2}{15}+\frac{3}{10}}$ **8.** $\dfrac{\frac{3}{4}+\frac{1}{6}}{\frac{3}{8}-\frac{1}{6}}$

The same two techniques are used to simplify rational expressions.

Example: Simplify $\dfrac{\frac{4}{x}+\frac{1}{x-1}}{\frac{1}{x-1}-\frac{1}{x}}$ by first simplifying numerator and denominator.

Let us simplify the numerator (that is, $\frac{4}{x}+\frac{1}{x-1}$). The denominators are x and $(x-1)$, so the LCD of these denominators is $x(x-1)$. Therefore,

$$\frac{4}{x}+\frac{1}{x-1}=\frac{4\,{\color{red}(x-1)}}{x\,{\color{red}(x-1)}}+\frac{1\cdot {\color{red}x}}{{\color{red}x}\,(x-1)}=\frac{4(x-1)+x}{x(x-1)}=\frac{5x-4}{x(x-1)}.$$

Next let us simplify the denominator (that is, $\dfrac{1}{x-1}-\dfrac{1}{x}$). The denominators are $(x-1)$ and x, so the LCD of these denominators is $x(x-1)$. Therefore,

$$\frac{1}{x-1}-\frac{1}{x}=\frac{1\,{\color{red}x}}{{\color{red}x}(x-1)}-\frac{1\,{\color{red}(x-1)}}{x{\color{red}(x-1)}}=\frac{x-(x-1)}{x(x-1)}=\frac{x-x+1}{x(x-1)}=\frac{1}{x(x-1)}.$$

Finally let us use the simplified forms of the numerator and the denominator to simplify the big fraction:

$$\frac{\frac{5x-4}{x(x-1)}}{\frac{1}{x(x-1)}}=\frac{5x-4}{x(x-1)}\cdot\frac{x(x-1)}{1}=\frac{(5x-4)x(x-1)}{x(x-1)}=\frac{(5x-4)\cancel{x}\cancel{(x-1)}}{\cancel{x}\cancel{(x-1)}}=5x-4.$$

Now let us do it using the other method:

Example: Simplify $\dfrac{\frac{4}{x}+\frac{1}{x-1}}{\frac{1}{x-1}-\frac{1}{x}}$ by multiplying numerator and denominator of the big fraction by the LCD.

The denominators of both the numerator and denominator are x and $(x-1)$, so the LCD is $x(x-1)$. Let us multiply numerator and denominator of the big fraction by $\frac{x(x-1)}{1}$:

$$\frac{\frac{4}{x}+\frac{1}{x-1}}{\frac{1}{x-1}-\frac{1}{x}}=\frac{\frac{x(x-1)}{1}\left(\frac{4}{x}+\frac{1}{x-1}\right)}{\frac{x(x-1)}{1}\left(\frac{1}{x-1}-\frac{1}{x}\right)}.$$

Let us simplify the numerator:

$$\frac{x(x-1)}{1}\left(\frac{4}{x}+\frac{1}{x-1}\right)=\frac{4x(x-1)}{x}+\frac{x(x-1)}{x-1}=\frac{4\cancel{x}(x-1)}{\cancel{x}}+\frac{x\cancel{(x-1)}}{\cancel{x-1}}=4(x-1)+x=4x-4+x=5x-4.$$

Now the denominator:

$$\frac{x(x-1)}{1}\left(\frac{1}{x-1}-\frac{1}{x}\right)=\frac{x(x-1)}{x-1}-\frac{x(x-1)}{x}=\frac{x\cancel{(x-1)}}{\cancel{x-1}}-\frac{\cancel{x}(x-1)}{\cancel{x}}=x-(x-1)=x-x+1=1.$$

Thus, the big fraction simplifies to

$$\frac{\dfrac{4}{x}+\dfrac{1}{x-1}}{\dfrac{1}{x-1}-\dfrac{1}{x}}=\frac{5x-4}{1}=5x-4.$$

It is quite amazing that such a complicated expression can be simplified so much!

Some of these examples can get quite complicated. It is best to do it in small parts, otherwise it is very messy.

Here is another example:

Example: Simplify $\dfrac{\dfrac{3}{x}+\dfrac{2}{x-2}}{\dfrac{3}{x-2}-\dfrac{1}{x-1}}$ by first simplifying numerator and denominator of the big fraction.

Let us simplify the numerator (that is, $\frac{3}{x}+\frac{2}{x-2}$).

The denominators are x and $(x-2)$, so the LCD of these denominators is $x(x-2)$. Therefore,

$$\frac{3}{x}+\frac{2}{x-2}=\frac{3(x-2)}{x(x-2)}+\frac{2x}{x(x-2)}=\frac{3(x-2)+2x}{x(x-2)}=\frac{3x-6+2x}{x(x-2)}\frac{5x-6}{x(x-2)}.$$

Next let us simplify the denominator (that is, $\frac{3}{x-2}-\frac{1}{x-1}$).

The denominators are $(x-1)$ and $(x-2)$, so the LCD of these denominators is $(x-1)(x-2)$. Therefore,

$$\frac{3}{x-2}-\frac{1}{x-1}=\frac{3(x-1)}{(x-1)(x-2)}-\frac{1(x-2)}{(x-1)(x-2)}=\frac{3(x-1)-(x-2)}{(x-1)(x-2)}=\frac{3x-3-x+2}{(x-1)(x-2)}=\frac{2x-1}{(x-1)(x-2)}.$$

Finally let us use the simplified forms of the numerator and the denominator to simplify the big fraction:

$$\frac{\dfrac{3}{x}+\dfrac{2}{x-2}}{\dfrac{3}{x-2}-\dfrac{1}{x-1}}=\frac{\dfrac{5x-6}{x(x-2)}}{\dfrac{2x-1}{(x-1)(x-2)}}=\frac{5x-6}{x(x-2)}\cdot\frac{(x-1)(x-2)}{2x-1}=\frac{(5x-6)(x-1)\cancel{(x-2)}}{x\cancel{(x-2)}(2x-1)}=\frac{(5x-6)(x-1)}{x(2x-1)}$$

Example: Simplify $\dfrac{\dfrac{3}{x}+\dfrac{2}{x-2}}{\dfrac{3}{x-2}-\dfrac{1}{x-1}}$ by multiplying by the LCD.

The denominators are x, $(x-1)$ and $(x-2)$, so the LCD of the denominators is the product of all three (since they all have exponent one). That is, the LCD is $x(x-1)(x-2)$.

Now let us multiply the numerator and denominator of the big fraction by $\dfrac{x(x-1)(x-2)}{1}$:

$$\frac{\dfrac{3}{x}+\dfrac{2}{x-2}}{\dfrac{3}{x-2}-\dfrac{1}{x-1}}=\frac{\dfrac{x(x-1)(x-2)}{1}\left(\dfrac{3}{x}+\dfrac{2}{x-2}\right)}{\dfrac{x(x-1)(x-2)}{1}\left(\dfrac{3}{x-2}-\dfrac{1}{x-1}\right)}.$$

Let us simplify the numerator first:

$$\frac{x(x-1)(x-2)}{1}\left(\frac{3}{x}+\frac{2}{x-2}\right)=\frac{3x(x-1)(x-2)}{x}+\frac{2x(x-1)(x-2)}{x-2}=\frac{3\cancel{x}(x-1)(x-2)}{\cancel{x}}+\frac{2x(x-1)\cancel{(x-2)}}{\cancel{x-2}}$$

$$=3(x-1)(x-2)+2x(x-1)=(x-1)(3(x-2)+2x)=(5x-6)(x-1).$$

Now let us do the denominator:

$$\frac{x(x-1)(x-2)}{1}\left(\frac{3}{x-2}-\frac{1}{x-1}\right)=\frac{3x(x-1)(x-2)}{x-2}-\frac{x(x-1)(x-2)}{x-1}=\frac{3x(x-1)\cancel{(x-2)}}{\cancel{x-2}}-\frac{x\cancel{(x-1)}(x-2)}{\cancel{x-1}}.$$

$$=3x(x-1)-x(x-2)=2x^2-x=x(2x-1)$$

Therefore, putting it all together, we have

$$\frac{\dfrac{3}{x}+\dfrac{2}{x-2}}{\dfrac{3}{x-2}-\dfrac{1}{x-1}}=\frac{(5x-6)(x-1)}{x(2x-1)}.$$

You will not have to do examples like the last one in this course, it is too long! But it is good to see a difficult example.

<u>Practice exercises</u>: Simplify the following by first simplifying the numerator and denominator of the big fraction.

9. $\dfrac{\dfrac{1}{x}+\dfrac{1}{x^2}}{\dfrac{1}{x}-\dfrac{1}{x^2}}$ **10.** $\dfrac{\dfrac{2}{x}+\dfrac{2}{y}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}}$ **11.** $\dfrac{4+\dfrac{4}{x-5}}{\dfrac{1}{x-5}+\dfrac{x}{4}}$

12. $\dfrac{\dfrac{1}{x}+\dfrac{x}{y}}{\dfrac{y}{x}-\dfrac{1}{y}}$ **13.** $\dfrac{x-\dfrac{2x}{x+3}}{\dfrac{1}{x+3}+\dfrac{1}{x-3}}$

<u>Practice exercises</u>: Simplify the following by Simplify the following by multiplying by the LCD.

14. $\dfrac{\dfrac{1}{x}+\dfrac{1}{x^2}}{\dfrac{1}{x}-\dfrac{1}{x^2}}$ **15.** $\dfrac{\dfrac{2}{x}+\dfrac{2}{y}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}}$ **16.** $\dfrac{4+\dfrac{4}{x-5}}{\dfrac{1}{x-5}+\dfrac{x}{4}}$

17. $\dfrac{\dfrac{1}{x}+\dfrac{x}{y}}{\dfrac{y}{x}-\dfrac{1}{y}}$ **18.** $\dfrac{x-\dfrac{2x}{x+3}}{\dfrac{1}{x+3}+\dfrac{1}{x-3}}$