6.2. Factor trinomials Professor Luis Fernández

Factoring trinomials of the form $x^2 + bx + c$

Recall: when we multiply, for example, $(x+2)(x+3)$, we get

$$(x+2)(x+3) = x^2 + 3x + 2x + 6 = x^2 + 5x + 6.$$

Notice that the middle coefficient is $5 = 2 + 3$ and the constant term is $6 = 2 \cdot 3$. We can do this in general: when we multiply $(x+m)(x+n)$, where m and n are any numbers, we get

$$(x+m)(x+n) = x^2 + nx + mx + mn = x^2 + (m+n)x + mn,$$

that is, the middle term is the sum of m and n and the constant term is the product of m and n.

Now let us do this backwards. To factor a polynomial of the form $x^2 + bx + c$ we want to find two numbers m and n such that

- $m + n = b$.
- $m \cdot n = c$.

Then we can immediately factor the polynomial: $x^2 + bx + c = (x+m)(x+n)$.

Example: To factor $x^2 + 7x + 12$, we would want two numbers m and n that add to 7 and multiply to 12. This is easy to find using trial and error: 3 and 4 will work. Thus we have

$$x^2 + 7x + 12 = (x+3)(x+4).$$

It is important to check that this is correct: $(x+3)(x+4) = x^2 + 4x + 3x + 12 = x^2 + 7x + 12$. It is correct.

Example: To factor $x^2 - x - 12$, we would want two numbers m and n that add to -1 and multiply to -12. Because $m \cdot n = -12$, which is negative, it must be that one of the numbers is positive and the other is negative. Then we can try all the numbers that multiply to -12, and we quickly find that the ones we want are -4 and 3 (because $-4 + 3 = -1$). Thus we have

$$x^2 - x - 12 = (x+3)(x-4).$$

It is important to check that this is correct: $(x+3)(x-4) = x^2 - 4x + 3x - 12 = x^2 - x - 12$. It is correct.

Sometimes it is not so straightforward to find the two numbers. In that case, the best solution is to try all the possibilities of numbers m and n such that $m \cdot n = c$. For example

Example: Factor $x^2 + 5x - 24$. We want m and n such that $m \cdot n = -24$ and $m + n = 5$. Let us list in an organized way all the possible values of m and n such that $m \cdot n = -24$ until we find those such that $m + n = 5$:

m	n	$m+n$
1	-24	-23
-1	24	23
2	-12	-11
-2	12	11
3	-8	-5
-3	8	5

We found them! $(-3) \cdot 8 = -24$ and $(-3) + 8 = 5$. Therefore

$$x^2 + 5x - 24 = (x+8)(x-3).$$

Check: $(x+8)(x-3) = x^2 - 3x + 8x - 24 = x^2 + 5x - 24$.

Practice exercises: Factor the following trinomials.

1. $x^2 - 6x + 8 =$

2. $x^2 + 9x + 18 =$

3. $x^2 - 5x + 6 =$

4. $x^2 - 3x - 10 =$

5. $x^2 + 4x - 12 =$

6. $x^2 - 7x - 8 =$

7. $x^2 + 9x + 14 =$

8. $x^2 + 9x + 8 =$

9. $x^2 - x - 6 =$

10. $x^2 + x - 2 =$

11. $x^2 + 10x + 16 =$

12. $x^2 + 15x + 36 =$

Factoring trinomials of the form $\boldsymbol{ax^2 + bx + c}$. The "$ac$" method

What if the coefficient of x^2 is not 1 as before? Then there is a method that combines the previous method with factoring by grouping. It works as follows:

- Find two numbers m and n such that $m + n = b$ and $m \cdot m = a \cdot c$.
- Since $b = m + n$, we can rewrite the polynomial as $ax^2 + mx + nx + c$.
- Factor $ax^2 + mx + nx + c$ by grouping.

Example: Factor $2x^2 + 9x + 9$. Here we have $a = 2$, $b = 9$, $c = 9$. Thus, $a \cdot c = 18$. We want m and n such that $m \cdot n = 18$ and $m + n = 9$. A bit of trying gives that 3 and 6 works. Thus we can rewrite the polynomial as $2x^2 + 3x + 6x + 9$. Now we factor by grouping:

$$2x^2 + 3x + 6x + 9 = x(2x + 3) + 3(2x + 3) = (x + 3)(2x + 3).$$

Let us check: $(x + 3)(2x + 3) = 2x^2 + 3x + 6x + 9 = 2x^2 + 9x + 9$.

Example: Factor $5x^2 - 9x - 2$. Here we have $a = 5$, $b = -9$, $c = -2$. Thus, $a \cdot c = -10$. We want m and n such that $m \cdot n = -10$ and $m + n = -9$. A bit of trying gives that -10 and 1 works. Thus we can rewrite the polynomial as $5x^2 - 10x + x - 2$. Now we factor by grouping:

$$5x^2 - 10x + x - 2 = 5x(x - 2) + (x - 2) = (5x + 1)(x - 2).$$

Let us check: $(5x + 1)(x - 2) = 5x^2 - 10x + x - 2 = 5x^2 - 9x - 2$.

Practice exercises: Factor the following trinomials.

13. $5x^2 + 21x + 4 =$

14. $8x^2 + 25x + 3 =$

15. $6x^2 + x - 15 =$

16. $2x^2 - 27x - 45 =$

17. $5x^2 - 9x + 4 =$

18. $4x^2 + 16x + 15 =$