5.2. Properties of exponents and scientific notation Professor Luis Fernández

<u>Recall</u>: The expression a^k means $a \cdot a \cdot \cdots \cdot a$ (k times) for example, $5^3 = 5 \cdot 5 \cdot 5 = 25 \cdot 5 = 125$.

In order to manipulate expressions with variables and exponents, it is important to understand the rules that can be used. These are as follows:

1. Product Property	$a^m \cdot a^n = a^{m+n}$.	5. Quotient to a Power Property	$\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$
2. Quotient Property	$\dfrac{a^m}{a^n} = a^{m-n}$	6. Zero Exponent Definition	$a^0 = 1$
3. Power Property	$(a^m)^n = a^{m \cdot n}$	7. Definition of Negative Exponent	$a^{-n} = \dfrac{1}{a^n}$
4. Product to a Power	$(ab)^m = a^m b^m$	8. Property of Negative Exponents	$\left(\dfrac{a}{b}\right)^{-m} = \left(\dfrac{b}{a}\right)^m$

<u>Examples</u>:

1. Product property:
$$x^3 \cdot x^5 = x^{3+5} = x^8. \qquad x^7 \cdot x^{-4} = x^{7+(-4)} = x^3. \qquad x^3 \cdot x^5 \cdot x^{-9} = x^{3+5+(-9)} = x^{-1}.$$

2. Quotient property:
$$\frac{x^9}{x^5} = x^{9-5} = x^4. \qquad \frac{x^7}{x^{-4}} = x^{7-(-4)} = x^{11}. \qquad \frac{x^6}{x^5} = x^1 = x. \qquad \frac{x^4}{x^9} = x^{-5}.$$

3. Power property:
$$(x^3)^4 = x^{3 \cdot 4} = x^{12}. \qquad (x^7)^{-4} = x^{7 \cdot (-4)} = x^{-28}. \qquad (x^3)^0 = x^{3 \cdot 0} = x^0.$$

4. Product to a power:
$$(xy)^4 = x^4 y^4. \qquad = (xyz)^{-4} = x^{-4} y^{-4} z^{-4}. \qquad (xy)^0 = x^0 y^0.$$

5. Quotient to a power property:
$$\left(\frac{x}{y}\right)^4 = \frac{x^4}{y^4}. \qquad \left(\frac{x}{z}\right)^{-4} = \frac{x^{-4}}{z^{-4}}. \qquad \left(\frac{x}{y}\right)^0 = \frac{x^0}{y^0}.$$

6. Zero exponent definition:
$$y^0 = 1 \qquad \left(\frac{x}{y}\right)^0 = 1. \qquad (xy)^0 = 1$$

7. Definition of negative exponent:
$$x^{-4} = \frac{1}{x^4} \qquad \left(\frac{x}{y}\right)^{-3} = \frac{1}{\left(\frac{x}{y}\right)^3} \qquad (xy)^{-10} = \frac{1}{(xy)^{10}}.$$

8. Property of negative exponent:
$$\left(\frac{x}{y}\right)^{-4} = \left(\frac{y}{x}\right)^4. \qquad \left(\frac{1}{x}\right)^{-3} = \left(\frac{x}{1}\right)^3 = x^3 \qquad \left(\frac{x}{y}\right)^{-(-4)} = \left(\frac{y}{x}\right)^{-4} = \left(\frac{x}{y}\right)^{-4}$$

<u>Practice exercises</u>: Use all the properties above to simplify and rewrite the following so that the final expression has only <u>positive exponents</u>.

1. $x^5 x^3 =$

2. $\dfrac{x^5}{x^3} =$

3. $(xy)^7$

4. $\dfrac{x^3}{x^9} =$

5. $\left(\dfrac{x}{y}\right)^3 =$

6. $(xy)^0$

7. $\left(\dfrac{x^5}{x^3}\right)^0 =$

8. $\left(\dfrac{x}{y}\right)^{-5} =$

9. $(x^3)^{-2} =$

<u>Practice exercises</u>: Use the properties above to simplify the following expressions so that the final expression has only <u>positive exponents</u>.

10. $(x^3y^{-6}z^5)(8x^{-3}yz^4) =$

11. $\dfrac{x^5y^3z^{-2}}{x^{-2}y^5z^{-3}} =$

12. $\left(\dfrac{12x^2y^{-3}}{4x^{-5}}\right)^{-2} =$

13. $(x^{-2}y^{-4}z^2)(8x^{-1}y^2z^3) =$

14. $\dfrac{x^{-5}y^3z^{-2}}{x^2y^3z^{-2}} =$

15. $\left(\dfrac{3x^4y^{-1}}{9x^4y^{-1}}\right)^2 =$

An intutive explanation of the properties

1. Product property: Note that this makes sense: for example, $x^3 \cdot x^4 = \underbrace{x \cdot x \cdot x}_{\text{3 times}} \cdot \underbrace{x \cdot x \cdot x \cdot x}_{\text{4 times}} = x^7$. In general,

$$x^3 \cdot x^4 = \underbrace{x \cdot x \cdots x}_{m \text{ times}} \cdot \underbrace{x \cdot x \cdots x}_{n \text{ times}} = x^{m+n}.$$

2. Quotient property: Recall that if we have the same factor in the numerator and denominator of a fraction, we can cancel it out. For example,

$$\frac{3 \cdot 7}{3 \cdot 5} = \frac{\cancel{3} \cdot 7}{\cancel{3} \cdot 5} = \frac{7}{5}.$$

So if we have, for example, $\dfrac{x^5}{x^3}$, then

$$\frac{x^5}{x^3} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = \frac{x^2}{1} = x^2.$$

Note that if the exponent in the numerator is smaller than the exponent in the denominator, then we would get 1 in the numerator. For example,

$$\frac{x^2}{x^6} = \frac{x \cdot x}{x \cdot x \cdot x \cdot x \cdot x \cdot x} = \frac{\cancel{x} \cdot \cancel{x}}{\cdot \cancel{x} \cancel{x} \cdot x \cdot x \cdot x \cdot x} = \frac{1}{x^4}.$$

On the other hand, according to the formula, this is.

$$\frac{x^2}{x^6} = x^{2-6} = x^{-4}.$$

Therefore it should be that $\dfrac{1}{x^4} = x^{-4}$. This motivates definition 7: x^{-n} is defined as $\dfrac{1}{x^n}$.

It also explains definition 6. For example,

$$\frac{x^2}{x^2} = \frac{\cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x}} = 1.$$

On the other hand,

$$\frac{x^2}{x^2} = x^{2-2} = x^0,$$

and therefore it must be that $x^0 = 1$, which is definition 6.

3. Power property. It is easy to see with an example why this works:

$$(x^2)^3 = \underbrace{x^2 \cdot x^2 \cdot x^2}_{\text{3 times}} = \underbrace{\underbrace{x \cdot x}_{\text{2 times}} \cdot \underbrace{x \cdot x}_{\text{2 times}} \underbrace{x \cdot x}_{\text{2 times}}}_{\text{3 times}} = x^{2\cdot3} = x^6.$$

In general, we would be multiplying x^m n times, so we would have a total of $m \cdot n$ factors.

4. Product to power property. If we do, for example, $(xy)^3$ we have

$$(xy)^3 = x \cdot y \cdot x \cdot y \cdot x \cdot y = x \cdot x \cdot x \cdot y \cdot y \cdot y = x^3 y^3.$$

Scientific Notation

Remember: A number is in scientific notation if it has only one digit in the whole part, and this digit is not zero.

Examples: 23.45 is not in scientific notation because it has two digits (2 and 3) in the whole part.

4.45004 is in scientific notation because the whole number part consists only of a nonzero digit.

0.45 (or .45) is not in scientific notation because the whole number part consists only of a 0.

Exercises: Which of these numbers are written in scientific notation?

16.	0.23	**17.**	2.55	**18.**	45.34
19.	123.23	**20.**	24.55	**21.**	0.004

Remember: Multiplying by a *positive* power of 10, say 10^n, amounts to *moving the decimal point n spaces to the right, adding 0s at the right of the number if necessary.*

Examples: $5.6352 \cdot 10^3 = 5635.2$ (moved the decimal point 3 spaces to the right);

$1.62 \cdot 10^5 = 162000$ (moved the decimal point 2 spaces to the right and then added 3 zeros at the right).

Likewise, Multiplying by a *negative* power of 10, say 10^{-n}, amounts to *moving the decimal point n spaces to the left, adding 0s at the left of the number if necessary.*

Examples: $3623.52 \cdot 10^{-3} = 3.62352$ (moved the decimal point 3 spaces to the left);

$1.62 \cdot 10^{-3} = 0.00162$ (moved the decimal point 1 space to the left and then added 2 zeros on the left, before the decimal point).

Exercises: Multiply

22.	$445 \times 10^3 =$	**23.**	$42965 \times 10^4 =$	**24.**	$153.5 \times 10^3 =$
25.	$345.654 \times 10^{-5} =$	**26.**	$179.65 \times 10^{-4} =$	**27.**	$0.05 \times 10^{-4} =$
28.	$215.34 \times 10^{-1} =$	**29.**	$423653 \times 10^{-7} =$	**30.**	$0.003 \times 10^{-1} =$

Exercises: The following numbers are written in scientific notation. Write them in usual notation.

31.	$2.3 \times 10^3 = 2300$ (example)	**32.**	$2.55 \times 10^5 =$	**33.**	$4.534 \times 10^{-4} =$
34.	$2.3 \times 10^{-3} =$	**35.**	$1.23 \times 10^4 =$	**36.**	$8.004 \times 10^{-4} =$
37.	$1.76 \times 10^8 =$	**38.**	$7.25 \times 10^{-6} =$	**39.**	$2.418 \times 10^{-1} =$

Remember: To write a number in scientific notation,

- Move the decimal place to the right or to the left until only one nonzero digit appears in the whole part (before the decimal point).
- Count how many places you moved the decimal point (say k places) and uultiply the number by 10^k if you moved the point to the left and by 10^{-k} if you moved the point to the right.

NOTE: To remember which sign of the exponent you need to use, think that when you moved the decimal point,

- If you made the number smaller, you need to multiply by a positive power of 10 to compensate.
- If you made the number greater, you need to multiply by a negative power of 10 to compensate.

Examples: $465.13 = 4.6513 \times 10^2$ (positive power because 4.6513 is smaller than 465.13).

Exercises: Write the following numbers in scientific notation.

40.	$23.3 =$ (example)	**41.**	$20355 =$	**42.**	$534.44 =$
43.	$0.00003 =$	**44.**	$1.45355 =$	**45.**	$5449994.34 =$
46.	$0.001001 =$	**47.**	$255443.234 =$	**48.**	$50000000 =$
49.	$24.54 \times 10^4 =$	**50.**	$322 \cdot 10^{-3} =$	**51.**	$534.44 \times 10^{34} =$

Multiplication and division in scientific Notation

Remember: To multiply two numbers that are written in scientific notation,

1. Multiply the number parts and the powers of 10 parts separately. Remember that to multiply the powers of 10 you only need to add the exponents.

2. Write the result of multiplying the number parts in scientific notation (if it is not in scientific notation already).

3. If you get an extra power of 10 from the previous item, multiply it with the other powers of 10.

Example: $(3 \times 10^4) \cdot (2 \times 10^3) = (3 \times 3) \times (10^4 \times 10^3) = 9 \times 10^7$.

Example: $(3.1 \times 10^{-3}) \cdot (5 \times 10^6) = (3.1 \times 5) \times (10^{-3} \times 10^6) = 15.5 \times 10^{-3+6} = 1.55 \times 10^1 \times 10^3 = 1.55 \times 10^4$.

NOTE: It is easy to forget to write the final result in scientific notation. Make sure you do step 2.

Exercises: Multiply and write the product in scientific notation.

52. $(2 \times 10^3) \cdot (3 \times 10^6) =$ **53.** $(4.5 \times 10^5) \cdot (2 \times 10^{12}) =$ **54.** $(6 \times 10^{-4}) \cdot (5 \times 10^{13}) =$

55. $(9 \times 10^{-7}) \cdot (3 \times 10^{-12}) =$ **56.** $(2 \times 10^8) \cdot (4.2 \times 10^{-10}) =$ **57.** $(3 \times 10^{-8}) \cdot (5 \times 10^2) =$

Remember: To divide two numbers that are written in scientific notation,

1. Divide the number parts and the powers of 10 parts separately. Remember that to divide the powers of 10 you only need to subtract the exponents of the denominator from the exponent in the numerator.

2. Write the result of dividing the number parts in scientific notation (if it is not in scientific notation already).

3. If you get an extra power of 10 from the previous item, multiply it with the other powers of 10.

Example: $\dfrac{6 \times 10^4}{2 \times 10^3} = \dfrac{6}{2} \times \dfrac{10^4}{10^3} = 2 \times 10^{4-3} = 2 \times 10^1$.

Example: $\dfrac{2 \times 10^{-3}}{4 \times 10^6} = \dfrac{2}{4} \times \dfrac{10^{-3}}{10^6} = 0.5 \times 10^{-3-6} = 0.5 \times 10^{-9} = 5 \times 10^{-1} \times 10^{-9} = 5 \times 10^{-10}$.

Exercises: Divide and write the product in scientific notation.

58. $\dfrac{4.5 \times 10^3}{3 \times 10^2} =$ **59.** $\dfrac{1.4 \times 10^5}{2 \times 10^2} =$ **60.** $\dfrac{6.3 \times 10^{-4}}{3 \times 10^{13}} =$

61. $\dfrac{3 \times 10^{-7}}{6 \times 10^{-12}} =$ **62.** $\dfrac{4 \times 10^8}{2 \times 10^{-10}} =$ **63.** $\dfrac{3 \times 10^{-8}}{2 \times 10^2} =$