5.1. Add and subtract polynomials Professor Luis Fernández

Some definitions

- **Terms:** In a mathematical expression, the *terms* are the parts of the expression separated by a + or a −. For example, in the expression $2x - 3y^4 + 6x - 4$ there are 4 terms:

$$\boxed{2x}\,\boxed{-3y^4} + \boxed{6x}\,\boxed{-4}.$$

NOTICE that each term is taken with its sign. That is, the second term in the last expression is $-3y^4$.

Each term can have any form, the point is that it is separated by the rest by + or −. For example, the terms in the expression $3(x+2) + 3y + \frac{3x+1}{4}$ are

$$\boxed{3(x+2)} + \boxed{3y} + \boxed{\frac{3x+1}{4}},$$

even though some of the terms themselves have + or − inside.

Exercises: Circle the terms in each of the following expressions.

1. $3x + 5y - 6$ **2.** $x^2 - 5x + 6$ **3.** $\dfrac{3x-2}{5} + 3x - 1$ **4.** $3(x + 2y - 1)$.

- **Polynomials:** A polynomial is a mathematical expression that can be written so that each term is either *a number* or *a number multiplying variables with or without exponents.* For example, $3x^3 - 5x^2 + x - 4$, $3xy^3 - 2xz - x^2 \cdot 4 + 6$, $\frac{1}{3}x^2y^3 + 3x$, 6, and 0, are all polynomials, whereas $\frac{3}{x^2+1} - 4$ is not a polynomial.

Exercises: Determine if the following are polynomials.

5. $3x + 5y - 6$ **6.** $x^2 - 5x + 6$ **7.** $\dfrac{3x-2}{5} + 3x - 1$ **8.** $\dfrac{3+x}{2+x} + 3x - 5$.

- **Monomial** A polynomial with only one term, like $2xy$, or $-3x^2y^5$, or -4.
- **Binomial** A polynomial with two terms, like $x^2 - 1$, or $-3x^2 + xyz^2$, or $-4 + 5y$.
- **Trinomial** A polynomial with three terms, like $x^2 + 3x + 2$, or $-3xy^5 + 1 - 7y$, or $-4 + 3x + 5y$.

Exercises: Classify the following in monomials, binomials, trinomials or neither.

9. $3x + 5y - 6$ **10.** $x^2 - 5x + 6$ **11.** $5x + 3$ **12.** $-3x^3y^2z^3$.

13. $3x + 5y$ **14.** x^2 **15.** $5x + 3 + 7y - 5xy$ **16.** $x^3 + 3x - 1$.

- **Coefficients:** In a term, the coefficient is the number that multiplies the variable(s), if any. For example, the coefficient of $-3xy^2$ is (-3), the coefficient of 5 is 5, and the coefficient of $\frac{-3}{5}x$ is $-\frac{3}{5}$.

Exercises: Circle each term and find the coefficient in each term of the following polynomial.

17. $4xy - 5x^4y^3 - 8 + 2x^3 + x^{12} \cdot 5 - x^2 \cdot 4 + x^{34}$

- **Degree of a term:** In a polynomial with only one variable, the degree is the exponent of the variable. Note that if there is no exponent written, then the exponent is 1, and if there is no variable, the exponent is 0. For example, the degree of $5x^3$ is 3, the degree of $-3x$ is 1, and the degree of 9 is 0.

Exercises: Find the coefficient and the degree for each term of the following polynomial.

18. $3x^7 + 5x^4 - 6 + 4x^3 + 6x^{12} - 5x^2 + x^{34}$

In order to work with polynomials more efficiently it is common to write the polynomial *with the degrees in descending order.* For example, we could write $4 + 5x - 3x^4 + x^2$, but it is easier to read if we write $-3x^4 + x^2 + 5x + 4$.

Exercises: Rewrite the following polynomials in descending order of degrees.

19. $x + x^3 - 6 - 5x^2$ **20.** $-4x + x^2 + 6$ **21.** $-3x + 4x^7 - 5x^2 + 6 - 17x^3 + 12x^8$.

- **Degree of a polynomial:** The maximum degree of all the terms. For example, the degree of $x + 5x^3 - 2$ is 3, the degree of $x + 5$ is 1, and the degree of 5 is 0.

- **Leading term:** The leading term is the term with highest degree. For example the leading term of $x + 5x^3 - 2$ is $5x^3$, the leading term of $x + 5$ is s, and the leading term of 5 is 5.
- **Constant term:** The term in a polynomial that has degree 0 (that is, it contains no variables). If no terms have degree 0 we say that the constant term is 0. For example, the constant term of $x + 5x^3 - 2$ is -2, the constant term of $x + 5$ is 5, and the constant term of $x^3 + 4x$ is 0.

  Exercises: Find the degree, the leading term, and the constant term of the following polynomials.

  22. $x + x^3 - 6 - 5x^2$ **23.** $-4x + x^2 + 6$ **24.** $-3x + 4x^7 - 5x^2 + 6 - 17x^3 + 12x^8$.
- **Like terms:** Two or more terms are *like terms* if they have the same variables with the same exponents (although the coefficient may be different). For example, $-5x^2$ and $3x^2$ are like terms, $3x^2y$ and $5yx^2$ are also like terms, whereas $3x$ and $2x^2$ are not like terms.

Adding polynomials: combining like terms

Like terms in a polynomial can be combined in a single term. For example, if we have $4x + 6x$ we can put the two terms together to get $10x$. Essentially, think that we have 4 copies of x together with 6 copies of x; so in total we have 10 copies of x.

To add polynomials you only have to combine like terms: first find which terms are like terms and then combine them. For example, in

$$(3x^2 + 4x - 5) + (x - 5x^3 - 6 + 4x^2),$$

$3x^2$ and $4x^2$ are like terms, $4x$ and x are like terms, and -5 and -6 are like terms, so we can combine them. It is easier to see if we remove the parenthesis and reorganize the polynomial in descending order:

$$(3x^2 + 4x - 5) + (x - 5x^3 - 6 + 4x^2) = -5x^3 + 3x^2 + 4x^2 + 4x + x - 5 - 6 = -5x^3 + 7x^2 + 5x - 11.$$

Practice exercises: Add the following polynomials and write the answer in descending order of degree.

25. $(x^2 + 3x - 2) + (3x^2 - 5x - 6)$ **26.** $(3x^3 - 4x - 2x^2) + (3x^2 - 6 + x)$

27. $(5x^3 - 2x - 1) + (3x^5 - 5^2)$ **28.** $(4x^2y + 3xy - 2) + (3xy - 5xy^2 + 6)$

29. $(3x - 4xy - z + x^2) + (-5x^2 + z - 3x)$ **30.** $(x^2 + 3x - 2) + (-3x^2 - 3x + 2)$

Subtracting polynomials

To subtract polynomials the only thing that has to be remembered is to start by distributing the "$-$" sign. Whenever there is a "$-$" in front of a parenthesis, it distributes to all the terms inside the parenthesis, that is

$$-(a + b) = -a - b.$$

For example, $-(x^2 + 3x - 5) = -x^2 - 3x + 5$. For example, $-(-x^3 - 2x^2 - 5x + 6) = x^3 + 2x^2 + 5x - 6$.

Once the "$-$" is distributed, just combine like terms as before.

Example: Simplify $(3x^2 + 3x - 5) - (3x^2 - 5x - 7)$. Start by distributing the $-$, and remove the parenthesis. Then combine like terms:

$$(3x^2 + 3x - 5) - (3x^2 - 5x - 7) = 3x^2 + 3x - 5 - 3x^2 + 5x + 7 = 8x + 2.$$

Practice exercises: Add the following polynomials and write the answer in descending order of degree.

31. $(x^2 + 3x - 2) - (3x^2 - 5x - 6)$ **32.** $(3x^3 - 4x - 2x^2) - (3x^2 - 6 + x)$

33. $(5x^3 - 2x - 1) - (3x^5 - 5^2)$ **34.** $(4x^2y + 3xy - 2) - (3xy - 5xy^2 + 6)$

35. $(3x - 4xy - z + x^2) - (-5x^2 + z - 3x)$ **36.** $x^2 + 3x - 2 - (-3x^2 - 3x + 2)$

37. $4 - (-9x^2 - x + 3)$ **38.** $(x^4 + x^2 - x) - (-5x^2 - 3)$

39. $2x^4 - 4xy^2 - z^2 - (-5x^2 + z^2 - 3y^2x)$ **40.** $-2 - (-8x^2 + 5x - 2)$