2.5. Solve linear inequalities. Professor Luis Fernández

Inequalities

Recall: An inequality is a mathematical expression containing one of the symbols $<$, $>$, $\leq$, or $\geq$. For example

$$2x - 3 < 5$$

is an inequality. Essentially they are equations where instead of an equal sign we have and inequality sign.

When we say that a variable is, for example, less than a number, we mean that any value of the variable less than that number is a correct solution. For example, for the inequality

$$x < 12,$$

any number less than 12 is a solution so, for example, 11, -1, -3.234, $4/5$ are all solutions (and there are infinitely more of course).

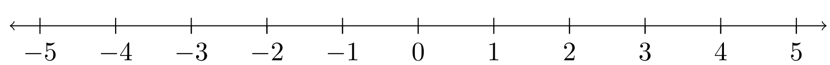
Exercise: Find 10 different solutions of the inequality $x \geq 3$.

Note that the order of inequalities can be reversed, being careful to write the opposite sign for the inequality. For example, $x < 5$ means the same thing as $5 > x$. In fact we do this in everyday language: saying "I am older than you" means the same as saying "you are younger than I".

Representing inequalities on the real line

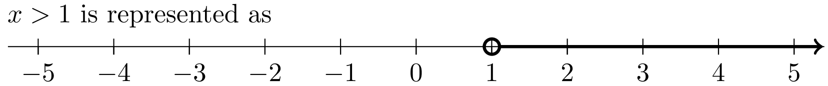
The set of numbers that are solutions of an inequality is generally infinite, so we *cannot* just make a list of all the solutions. Instead, there are different ways to represent these sets.

Recall: To represent the set of all the real numbers we use a line, called the *number line*:



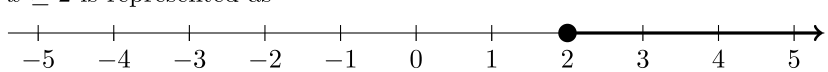
To represent the set of numbers greater than a given one, shade the part of the line above that number. For example:

- $x > 1$ is represented as



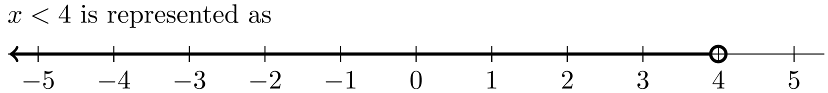
The "∘" at the point 1 means that the value 1 itself is not part of the set. If the point is part of the set, we put a filled dot:

- $x \geq 2$ is represented as

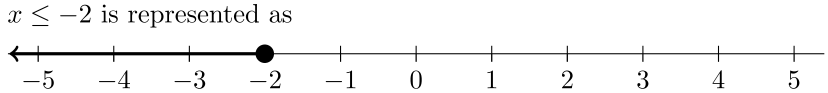


Likewise,

- $x < 4$ is represented as



- $x \leq -2$ is represented as



Practice exercises: Represent the following sets on the real line.

1. $x > -1$ **2.** $x \leq 0$ **3.** $x > 0$

4. $x < 3$ **5.** $x \geq -1.5$ **6.** $x \leq 2/3$

7. $1 < x < 3$ **8.** $-4 \leq x < -1.5$ **9.** $0 < x \leq 2/3$

Interval notation

Another way to represent the solution set of an inequality is using *interval notation*. It has the form "(lower number, higher number)". For example, the expression $(3, 6)$ means "all the numbers between 3 and 6, not including 3 or 6". To represent that the set goes up forever we use "∞", and to represent that the set goes down for ever, we use "$-\infty$".

Note: The lower number always has to go first!!

For example, $(3, 1)$ is NOT an interval, whereas $(1, 3)$ IS an interval.

To denote whether the point at the end is included in the interval or not, we use

- Square parenthesis "]" or "[" when the point is in the set.
- Round parenthesis ")" or "(" when the point is *not* in the set.

At ∞ or $-\infty$ we always use round parenthesis.

Example: The set of solutions of the equation $x > 4$ is written as $(4, \infty)$.

Example: The set of solutions of the equation $x \leq -3$ is written as $(-\infty, -3)$.

Practice exercises: Represent the solution sets of the following equations in interval notation.

10. $x > -1$ **11.** $x \leq 0$ **12.** $x > 0$

13. $x < 3$ **14.** $x \geq -1.5$ **15.** $x \leq 2/3$

16. $1 < x < 3$ **17.** $-4 \leq x < -1.5$ **18.** $0 < x \leq 2/3$

Solving inequalities

Inequalities are solved almost the same way as equations. The final goal is to arrive to an inequality of the kind "$x >$number", or "$x <$number", or "$x \geq$number", or "$x \leq$number". To get there, use the same techniques as for equations, with one change:

1. You can add **the same** number or expression to **both sides** of the inequality.

2. You can subtract **the same** number or expression from **both sides** of the inequality.

3a. You can multiply **both sides** of the inequality by the same **positive** number or expression.

3b. You can multiply **both sides** of the inequality by the same **negative** number or expression, but then **the inequality changes direction** (swap $>$ and $<$, or $\geq$ and $\leq$).

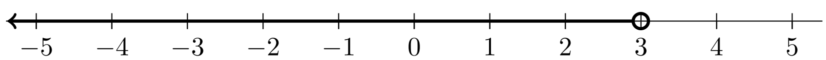
4. You can divide **both sides** of the inequality by the same **positive** number or expression.

4b. You can divide **both sides** of the inequality by the same **negative** number or expression, but then **the inequality changes direction** (swap $>$ and $<$, or $\geq$ and $\leq$).

<u>Example</u>: Solve the inequality $3x - 5 < 2x - 2$, and express the answer in both the real line and interval notation.

Start with $3x - 5 < 2x - 2$. Add 5 to both sides to get $3x < 2x + 3$.

Then subtract $2x$ from both sides to get $x < 3$. Therefore the solution set is $(-\infty, 3)$ or, on the real line,



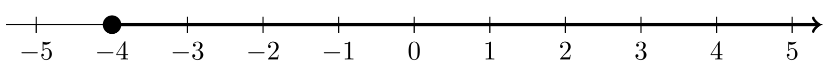
<u>Example</u>: Solve the inequality $3x - 12 \leq 7x + 4$. Express the answer in both the real line and interval notation.

Start with $3x - 12 \leq 7x + 4$. Add 12 to both sides to get $3x \leq 7x + 16$.

Then subtract $7x$ from both sides to get $-4x \leq 16$.

Finally divide both sides by -4 and **remember to swap the symbol of the inequality** to get $x \geq -4$.

Therefore the solution set is $[-4, \infty)$ or, on the real line,



<u>Practice exercises</u>: Solve the following inequalities and represent the answer both on the real line and in interval notation.

19. $2x + 7 > 15$	**20.** $5x - 4 < 16$	**21.** $3x - 5 < 12$
22. $6 - 2x \leq 14$	**23.** $-8 - 7x > -1$	**24.** $-5x + 7 > 12$
25. $6x - 5 < 2x - 13$	**26.** $x + 2 \geq 2 + 4x$	**27.** $4 \geq 2 + x$
28. $\frac{x}{5} + 6 < 9$	**29.** $\frac{5x}{2} \geq 15$	**30.** $\frac{-4x}{3} \leq -16$
31. $5(x - 1) + 3 \geq 5x - 2$	**32.** $-(x - 2) + 4 > 7 - x$	**33.** $2(x + 1) - 1 \geq 3 - x$
34. $6x - 2(x + 3) \geq -(4x + 6)$	**35.** $2 - (x + 1) < 5 - 2(x - 1)$	**36.** $-2(x - 2) + 2(x - 2) \leq 1$