8.8. Complex numbers. Professor Luis Fernández

Square roots of negative numbers.

Recall: A number to the square can never be negative: if the number is positive, the square is certainly positive, and if it is negative, the square is also positive (because negative times negative is positive).

This means that the square root of a negative number is not a real number.

We could, however, *extend* the kind of numbers that we use in order to make sense of square roots of negative numbers. This is not strange: in fact, *negative numbers* were invented to make sense of expressions like $5-7$. Then *rational numbers* were invented to make sense of expressions like $5 \div 3$. And then *irrational numbers* were invented to make sense of expressions like $\sqrt{2}$. Now we do it again and define the **complex numbers**.

We start by "making up" a number that equals the square root of -1. It is called the *imaginary unit*:

$$\boxed{\text{Define} \quad i = \sqrt{-1} \quad (\text{that is, } i \text{ is the "number" so that } i^2 = -1.)}$$

Using the imaginary unit i and the properties of square roots now we can make sense of the square root of any negative number. For example,

$\sqrt{-9} = \sqrt{9 \cdot (-1)} = \sqrt{9} \cdot \sqrt{-1} = 3 \cdot i.$

$\sqrt{-49} = \sqrt{49 \cdot (-1)} = \sqrt{49} \cdot \sqrt{-1} = 7i.$

$\sqrt{-5} = \sqrt{5 \cdot (-1)} = \sqrt{5} \cdot \sqrt{-1} = \sqrt{5} \cdot i$ (or $i\sqrt{5}$).

$\sqrt{-12} = \sqrt{12 \cdot (-1)} = \sqrt{12} \cdot \sqrt{-1} = \sqrt{4}\sqrt{3} \cdot i = 2i\sqrt{3}.$

Exercises: Write the following as $b \cdot i$ with b a whole number or a simplified radical expression.

1. $\sqrt{-25}$ **2.** $\sqrt{-36}$ **3.** $\sqrt{-100}$ **4.** $\sqrt{-4}$

5. $\sqrt{-7}$ **6.** $\sqrt{-13}$ **7.** $\sqrt{-20}$ **8.** $\sqrt{-50}$

Powers of the imaginary unit i

The imaginary unit i is defined so that $i^2 = -1$. What are i^3, i^4, i^5, etc? Using the rules of exponents these are easy to find:

$$\begin{aligned}
i^3 &= i^2 \cdot i = (-1) \cdot i = -i. \\
i^4 &= i^3 \cdot i = (-i) \cdot i = -i^2 = -(-1) = 1. \\
i^5 &= i^4 \cdot i = 1 \cdot i = i. \\
i^6 &= i^4 \cdot i^2 = 1 \cdot (-1) = -1. \\
i^7 &= i^4 \cdot i^3 = 1 \cdot (-i) = -i. \\
i^8 &= i^4 \cdot i^4 = 1 \cdot 1 = 1.
\end{aligned}$$

As you can see, we get a pattern that repeats every 4:

$$\begin{array}{cccc}
i^1 = i & i^2 = -1 & i^3 = -i & i^4 = 1 \\
i^5 = i & i^6 = -1 & i^7 = -i & i^8 = 1 \\
i^9 = i & i^{10} = -1 & i^{11} = -i & i^{12} = 1
\end{array}$$

Thus, if n is a multiple of 4, then $i^n = 1$. This gives an easy way to find any power of i as long as you remember that $i^2 = -1$ and $i^3 = -i$: for example $i^{27} = i^{24} \cdot i^3 = 1 \cdot (-i) = -i$.

Negative powers follow the same pattern. To find i^{-1} we can do:

$$i^{-1} = \frac{1}{i} = \frac{1 \cdot i}{i \cdot i} = \frac{i}{-1} = -i.$$

Likewise, $i^{-2} = -1$, $i^{-3} = i$ and $i^{-4} = 1$, and then the pattern is repeated as the exponent goes down.

Exercises:

9. Write the value of: $i^0, i^1, i^2, i^3, i^4, i^5, i^6, i^7, i^8, i^9, i^{10}, i^{11}, i^{12}, i^{13}, i^{14}, i^{15}, i^{16}$.

10. Write the value of: $i^0, i^{-1}, i^{-2}, i^{-3}, i^{-4}, i^{-5}, i^{-6}, i^{-7}, i^{-8}, i^{-9}, i^{-10}, i^{-11}, i^{-12}, i^{-13}, i^{-14}, i^{-15}, i^{-16}$.

Find the following powers of the imaginary unit i.

11. i^{40} **12.** i^{41} **13.** i^{17} **14.** i^{101} **15.** i^{-31}

Complex numbers

A *complex number* is a combination of a real number and another real number times the imaginary unit. That is, an expression of the form

$$a + ib$$

where a and b are real numbers.

- The *real part* of the complex number $a + ib$ is a, that is, the number that is not multiplying i.
- The *imaginary part* of the complex number $a + ib$ is b, that is, the number that is multiplying i.

  For example, the real part of $5 + 3i$ is 5, and the imaginary part is 3.

  The real part of $4 - 5i$ is 4, and the imaginary part is (-5).

  The real part of -7 is (-7), and the imaginary part is 0.

  The real part of $-8i$ is 0, and the imaginary part is (-8).

Exercises: Find the real and imaginary part of the following complex numbers.

16. $2 + 3i$ **17.** $4 - 5i$ **18.** $3 + i\sqrt{2}$ **19.** i

20. 1 **21.** $5i + 6$ **22.** $3i - 1$ **23.** $\frac{1}{3} + \frac{2i}{3}$

The Complex numbers can be added, subtracted, multiplied and divided. It is important to remember that i is just $\sqrt{-1}$, so all these operations are done exactly the same way as we did operations with radical expressions.

Addition and subtraction of complex numbers

Recall: To add or subtract radical expressions you only need to combine like radicals. For example,

$$(3 + 2\sqrt{5}) + (-2 + 4\sqrt{5}) = (3 - 2) + (2 + 4)\sqrt{5} = 1 + 6\sqrt{5}.$$

With complex numbers it is exactly the same: combine the real parts, combine the imaginary parts, and you are done.

Example: Add $(3 + 5i) + (4 - 2i)$.

$(3 + 5i) + (4 - 2i) = (3 + 4) + (5 - 2)i = 7 + 3i$.

Example: Subtract $(2 - 3i) - (-4 - 7i)$.

$(2 - 3i) - (-4 - 7i) = (2 - (-4)) + (-3 - (-7))i = 6 + 4i$.

Exercises: Add or subtract (as indicated) the following complex numbers.

24. $(2 + 3i) + (7 - 5i)$ **25.** $(4 - 5i) + (4 + 5i)$ **26.** $(3 + 6i) + (3 + 4i)$ **27.** $(3 + 2i) + (5 - 4i)$

28. $(2 + 3i) - (7 - 5i)$ **29.** $(4 - 5i) - (4 + 5i)$ **30.** $(3 + 6i) - (3 + 4i)$ **31.** $\left(\frac{1}{2} - \frac{5i}{3}\right) + \left(\frac{1}{3} + \frac{3i}{2}\right)$

32. $i - (1 - i)$ **33.** $4i - 3 + 8$ **34.** $(3i - 1) - (4 - 5i)$ **35.** $(7 - 4i) - (-5 - 6i)$

Multiplication of complex numbers

Again, multiplication of complex numbers is done exactly as with radical expressions: first distribute, then simplify powers of i and then combine like terms. For example

$$\begin{aligned}(4+3i)(5+2i) &= 4\cdot 5+4\cdot 2i+3i\cdot 5+(3i)\cdot(2i)\\ &= 20+8i+15i+6\cdot i^2\\ &= 20+8i+15i+6\cdot(-1) \quad \text{because } i^2=-1\\ &= 14+23\,i\end{aligned}$$

Another example:

$$\begin{aligned}(3-4i)(2+5i) &= 3\cdot 2+3\cdot 5i-4i\cdot 2-(4i)\cdot(5i)\\ &= 6+15i-8i-20\cdot i^2\\ &= 6+15i-8i-20\cdot(-1) \quad \text{because } i^2=-1\\ &= 26+7i\end{aligned}$$

Exercises: Multiply the following complex numbers.

36. $(2+3i)(7+5i)$ **37.** $(4-5i)(4+5i)$ **38.** $(3+6i)(3+4i)$ **39.** $i(2+3i)$

40. $-5(-2+3i)$ **41.** $(4-5i)(4+5i)$ **42.** $(3+4i)(3-4i)$ **43.** $\left(\frac{1}{2}-\frac{5i}{3}\right)\cdot\left(\frac{1}{3}+\frac{3i}{2}\right)$

44. $(2+i)(2i-7)$ **45.** $(5-3i)^2$ **46.** $(2+i)^2$ **47.** $(7-4i)^2$

Division of complex numbers

The goal of division of complex numbers is to take an expression like $\dfrac{2+3i}{5+2i}$ and write it in the form $a+bi$.

We already know how to do this with radicals: **we just need to multiply numerator and denominator by the conjugate of the denominator** (recall that the conjugate of a binomial is the same binomial with one of the signs changed). For example, since the conjugate of $(5+2i)$ is $(5-2i)$, we can do:

$$\begin{aligned}\frac{2+3i}{5+2i} &= \frac{(2+3i)(5-2i)}{(5+2i)(5-2i)}\\ &= \frac{10-4i+15i-6i^2}{5^2-(2i)^2}\\ &= \frac{10-4i+15i-6(-1)}{25-4i^2}\\ &= \frac{10-4i+15i+6}{25-4(-1)}\\ &= \frac{16+11i}{29}\end{aligned}$$

In the denominator of the last example we used the fact that $(x+y)(x-y)=x^2-y^2$. In fact, since y will always be of the form $b\cdot i$, this is always quite easy:

$$(a+bi)(a-bi)=a^2-(bi)^2=a^2-b^2i^2=a^2-b^2\cdot(-1)=a^2+b^2.$$

Thus, the denominator will always be the sum of the squares of the real and imaginary parts of the complex number.

Example: Divide: $\dfrac{1}{3+2i}$.

Multiply numerator and denominator by the conjugate of the denominator, which is $(3-2i)$, and notice that in the denominator we will get $(3+2i)(3-2i)=3^2+2^2=9+4=13$:

$$\frac{1}{3+2i}=\frac{1\cdot(3-2i)}{(3+2i)(3-2i)}=\frac{3-2i}{13}.$$

Example: Divide: $\dfrac{5-3i}{4+2i}$.

Multiply numerator and denominator by the conjugate of the denominator, which is $(4-2i)$, and notice that in the denominator we will get $(4+2i)(4-2i) = 4^2 + 2^2 = 16 + 4 = 20:$

$$\begin{aligned}
\frac{5-3i}{4+2i} &= \frac{(5-3i)\cdot(4-2i)}{(3+2i)(3-2i)} \\
&= \frac{20-10i-12i+6i^2}{20} \\
&= \frac{20-10i-12i+6(-1)}{20} \\
&= \frac{14-22i}{20} \\
&= \frac{\cancel{2}(7-11i)}{\cancel{20}_{10}} \\
&= \frac{7-11i}{10}
\end{aligned}$$

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Exercises: Divide the following complex numbers.

48. $\dfrac{1}{2+5i}$ **49.** $\dfrac{3}{i}$ **50.** $\dfrac{3+2i}{5-3i}$ **51.** $\dfrac{i}{2+3i}$

52. $\dfrac{1+i}{1-i}$ **53.** $\dfrac{4-5i}{4+5i}$ **54.** $\dfrac{3+4i}{1-4i}$ **55.** $\dfrac{i+7}{1+i}$