7.4. Solve rational equations Professor Luis Fernández

Rational equations

A *rational equation* is an equation involving rational expressions. For example,

$$\frac{3}{x}+5=\frac{2}{x} \qquad \frac{x-1}{x+3}+3=\frac{x+1}{x+3} \qquad \frac{x+1}{x-1}+\frac{x-1}{x}=0$$

are examples of rational equations.

A solution of a rational equation is a value of the variable that gives a true statement when it is substituted into the equation.

NOTE: There may be values of the variable for which a rational expression is not defined. Namely, the values of the variable that make any denominator $=0$. **These values cannot be solutions.** For example in the equation

$$\frac{x+1}{x-1}+\frac{x-1}{x}=0$$

the values $x=1$ and $x=0$ **cannot** be solutions because if you substitute them into the equation you get division by 0, which is not defined.

Solving rational equations

As in the previous section (complex rational expressions) there are two methods to solve rational equations. We will only study one of them.

The idea is to multiply both sides of the equation by the LCM of all the denominators (in both sides) so that all the denominators cancel out and then solve it as a polynomial equation.

Example: Solve $\dfrac{3}{x}+5=\dfrac{2}{x}$

The denominator in both sides is just x. Multiply both sides by x and simplify:

$$\frac{x}{1}\cdot\left(\frac{3}{x}+5\right)=\frac{x}{1}\cdot\frac{2}{x}.$$

Distribute to get

$$\frac{x}{1}\cdot\frac{3}{x}+\frac{x}{1}\cdot 5=\frac{x}{1}\cdot\frac{2}{x},$$

and cancel out common factors:

$$\frac{\cancel{x}}{1}\cdot\frac{3}{\cancel{x}}+5x=\frac{\cancel{x}}{1}\cdot\frac{2}{\cancel{x}}.$$

We get the equation $3+5x=2$, which is easy to solve: $3+5x=2 \ \rightarrow\ 5x=-1 \ \rightarrow\ x=-\dfrac{1}{5}$.

Finally check that the solution you got does not make any of the denominators in the original equation equal to 0. Therefore the solution is $x=-\dfrac{1}{5}$.

Summarizing, to solve a rational equation,

- Find the LCM of all the denominators in the equation.
- Multiply both sides of the equation by this LCM and simplify (all denominators should cancel).
- Solve the equation you get. Check that your answers make sense when they are substituted into the original equation.

Example: Solve $\dfrac{3}{x-3} + \dfrac{2}{x+3} = \dfrac{x}{x^2-9}$

First we find the LCD of the denominators. To do this, recall that we first need to factor them. $(x-3)$, $(x+3)$ cannot be factored further, and (x^2-9) is factored as $(x+3)(x-3)$. Thus the factors appearing in all the denominators are just $(x+3)$ and $(x-3)$, both with highest exponent 1. Therefore the LCD is $(x+3)(x-3)$.

Next, multiply both sides by $(x+3)(x-3)$:

$$\frac{(x+3)(x-3)}{1} \cdot \left(\frac{3}{x-3} + \frac{2}{x+3} \right) = \frac{(x+3)(x-3)}{1} \cdot \frac{x}{x^2-9}$$

Then distribute and simplify, noting that $x^2-9 = (x+3)(x-3)$:

$$\frac{(x+3)\cancel{(x-3)}}{1} \cdot \frac{3}{\cancel{x-3}} + \frac{\cancel{(x+3)}(x-3)}{1} \cdot \frac{2}{\cancel{x+3}} = \frac{\cancel{(x+3)}\cancel{(x-3)}}{1} \cdot \frac{x}{\cancel{(x+3)}\cancel{(x-3)}},$$

and we get $3(x+3) + 2(x-3) = x$ which is easy to solve:

$$\begin{aligned} 3(x+3) + 2(x-3) &= x \\ 3x + 9 + 2x - 6 &= x \\ 5x + 3 &= x \\ 4x &= -3 \\ x &= -\frac{3}{4}. \end{aligned}$$

The only values that make denominators equal to 0 in the original equation are 3 and -3, so our solution is not problematic. Therefore the solution is $x = -\dfrac{3}{4}$.

Practice exercises: Solve the following rational equations.

1. $1 - \dfrac{2}{x} = \dfrac{15}{x^2}$

2. $\dfrac{2}{x+1} + \dfrac{1}{x-1} = \dfrac{1}{x^2-1}$

3. $\dfrac{x-6}{x^2+3x-4} = \dfrac{2}{x+4} + \dfrac{7}{x-1}$

4. $\dfrac{x}{x+4} = \dfrac{32}{x^2-16} + 5$

5. $\dfrac{2}{x+7} - \dfrac{3}{x-3} = 1$

6. $\dfrac{x-10}{x^2-5x+4} = \dfrac{3}{x-1} - \dfrac{6}{x-4}$

7. $\dfrac{15}{x^2+x-6} - \dfrac{3}{x-2} = \dfrac{2}{x+3}$

8. $\dfrac{5}{x^2+2x-3} - \dfrac{3}{x^2+x-2} = \dfrac{1}{x^2+5x+6}$

9. $\dfrac{x}{5x-10} - \dfrac{5}{3x+6} = \dfrac{2x^2-19x+54}{15x^2-60}$