8.6. Radical equations Professor Luis Fernández

Radical equations.

An radical equation is an equations where the variable is in the radicand of a radical expression. For example,

$$\sqrt{x-3}+4=6 \qquad \sqrt[3]{x^2+3}-2=\sqrt{x+2} \qquad \text{and} \qquad \sqrt{x+3}-2=\sqrt{x+1}$$

are all radical equations.

To solve a radical equation one has to start by removing the radicals. The only way to remove an n^{th} root is by doing the opposite, that is, raising to the n^{th} power. This uses the property that we already know:

$$(\sqrt[n]{x})^n = x \text{ when } x \geq 0.$$

Let us see an example:

Example: Solve $\sqrt{3x-2}=5$.

Let us raise both sides to the square: $(\sqrt{3x-2})^2=5^2$. Since $(\sqrt{3x-2})^2=3x-2$, this gives $3x-2=25$. Finally, adding 2 to both sides and dividing both sides by 3 gives $x=9$.

Let us check the solution: the left hand side gives $\sqrt{3\cdot 9-2}=\sqrt{27-2}=\sqrt{25}=5$, which is equal to the right hand side. Thus, the solution is $x=9$.

Thus, to solve an equation where the variable is inside a square root, one has to isolate the radical on one side of the equation, raise both sides to the square to remove the radical, and then simplify and solve the resulting equation. Then it is essential to check the solution by substituting it into the original equation: the procedure of squaring both sides can give *extraneous* solutions that are not solutions after all (similar to what happened when we solved rational equations).

Example: Solve $\sqrt{5-x}+3=5$.

Let us isolate the square root by subtracting 3 from both sides, to get $\sqrt{5-x}=2$.

Then raise both sides to the square: $(\sqrt{5-x})^2=2^2$.

Since $(\sqrt{5-x})^2=5-x$, this gives $5-x=4$.

Finally, subtracting 5 from both sides and dividing both sides by -1 gives $x=1$.

Let us check the solution: the left hand side gives $\sqrt{5-1}+3=\sqrt{4}+3=2+3=5$, which is equal to the right hand side. Thus, the solution is $x=1$.

Note that some equations do not have solutions, as in the following example:

Example: Solve $\sqrt{x+5}+10=-1$.

Start by subtracting 10 from both sides to get $\sqrt{x+5}=-11$.

But now notice that the last expression cannot be true for any value of x because the square root of any number is always positive, so the left hand side of the equation is positive, but the left hand side is negative.

Therefore, the equation has no solution.

Example: Solve $\sqrt[3]{2x-3}+11=8$.

Let us isolate the square root by subtracting 11 from both sides, to get $\sqrt[3]{2x-3}=-3$.

Then raise both sides to the cube: $(\sqrt[3]{2x-3})^3=-27$.

Since $(\sqrt[3]{2x-3})^3=2x-3$, this gives $2x-3=-27$.

Finally, adding 3 to both sides and dividing both sides by 2 gives $x = -12$.

Let us check the solution: the left hand side gives $\sqrt[3]{2 \cdot (-12) - 3} + 11 = \sqrt[3]{-24-3} + 11 = \sqrt[3]{-27} + 11 = -3 + 11 = 8$, which is equal to the right hand side.

Therefore the solution is $x = -12$.

Exercises Solve the following equations.

1. $\sqrt{5x-6} = 8$ **2.** $\sqrt[3]{4x-1} = 3$ **3.** $\sqrt{x+3} = -5$

4. $\sqrt{2x-5} = 7$ **5.** $\sqrt[3]{2x+1} = 1$ **6.** $\sqrt{6x+1} + 4 = 8$

Things complicate a little when the variable is both inside and outside the radical. Let us see an example:

Example: Solve $\sqrt{x+10} + 2 = x$.

To isolate the radical, start by subtracting 2 from both sides to get $\sqrt{x+10} = x - 2$.

Then raise both sides to the square to get $(\sqrt{x+10})^2 = (x-2)^2$.

Simplifying the left hand side and expanding the right hand side we get $x + 10 = x^2 - 4x + 4$. (To simplify the right hand side, recall that $(a+b)^2 = a^2 + 2ab + b^2$.)

We obtain a quadratic equation, so we have to move everything to one side: subtract x and 10 from both sides to get: $0 = x^2 - 5x - 6$.

Then factor (and swap the sides if you want): $(x-6)(x+1) = 0$. This breaks into the equations $x - 6 = 0$, which gives $x = 6$, and $x + 1 = 0$, which gives $x = -1$.

Thus, the solutions are $x = 6$ and $x = -1$.

Let us check $x = 6$: The left hand side is $\sqrt{6+10} + 2 = \sqrt{16} + 2 = 4 + 2 = 6$. The right hand side is also 6. Therefore 6 is a solution.

Let us check $x = -1$: The left hand side is $\sqrt{-1+10} + 2 = \sqrt{9} + 2 = 3 + 2 = 5$. However, the right hand side is -1. Therefore -1 is **not** a solution.

Therefore, the only solution is $x = 6$.

Exercises Solve the following equations.

7. $\sqrt{x-6} = x - 6$ **8.** $\sqrt{x+4} = x - 2$ **9.** $\sqrt{x+1} = x - 1$

10. $\sqrt{x+25} - x = -5$ **11.** $\sqrt{x+3} + x = 9$ **12.** $\sqrt{2x+6} - x + 1 = 0$

When the original equation has more than one radical, one has to do the process above twice. For example,

Example: Solve $\sqrt{x} + 2 = \sqrt{x+16}$.

Since the radical on the right hand side is isolated, let us raise both sides to the square to get $(\sqrt{x}+2)^2 = (\sqrt{x+16})^2$

The left hand side gives $(\sqrt{x}+2)^2 = (\sqrt{x})^2 + 2 \cdot 2\sqrt{x} + 2^2 = x + 4 + 4\sqrt{x}$ (use $(a+b)^2 = a^2 + 2ab + b^2$).

The right hand side is just $(\sqrt{x+16})^2 = x + 16$. Therefore we get $x + 4 + 4\sqrt{x} = x + 16$.

But now notice that we still have a square root! So we have to do the same thing again: let us isolate the radical by subtracting x and 4 from both sides, and then dividing by 4.

We get $\sqrt{x} = 3$.

Raise both sides to the square to get $(\sqrt{x})^2 = 3^2$, or $x = 9$. Therefore the answer is $x = 9$.

Let us check the solution: the left hand side gives $\sqrt{9} + 2 = 3 + 2 = 5$. The right hand side gives $\sqrt{9+16} = \sqrt{25} = 5$. Since they are equal, the solution is $x = 9$.

Summarizing, to solve a radical equation,

- Isolate one of the radical terms on one side of the equation.
- Raise both sides of the equation to the power of the index.
- Are there any more radicals?
  - → If yes, repeat Step 1 and Step 2 again.
  - → If no, solve the new equation.
- Check the answer by substituting in the original equation.

Let us see a more complicated example.

Example: Solve $\sqrt{x+5} - \sqrt{x} = 1$.

Let us first isolate $\sqrt{x+5}$ on the left: add $\sqrt{x}$ to both sides to get $\sqrt{x+5} = \sqrt{x} + 1$.

Then raise both sides to the square: $(\sqrt{x+5})^2 = (\sqrt{x}+1)^2$.

The left hand side is $(\sqrt{x+5})^2 = x+5$.

To simplify the right hand side, recall that $(a+b)^2 = a^2 + 2ab + b^2$.

Thus the right hand side is $(\sqrt{x}+1)^2 = (\sqrt{x})^2 + 2 \cdot 1 \cdot \sqrt{x} + 1^2 = x + 2\sqrt{x} + 1$,

and the equation has been transformed into $x + 5 = x + 1 + 2\sqrt{x}$.

Now we need to isolate the remaining radical. Let us isolate $\sqrt{x}$ on the right hand side. To this end, subtract $x+1$ from both sides to get

$$4 = +2\sqrt{x},$$

and dividing by 2 we find

$$2 = \sqrt{x}.$$

Finally, squaring both sides we get $x = 4$. Let us check it:

The right hand side is $\sqrt{4+5} - \sqrt{4} = \sqrt{9} - \sqrt{4} = 3 - 2 = 1$, which is equal to the left hand side.

Thus, the solution is $x = 4$.

Exercises Solve the following equations.

13. $\sqrt{3x+7} = \sqrt{5x+1}$ **14.** $\sqrt[3]{8x-5} = \sqrt[3]{3x+5}$ **15.** $\sqrt{x+1} = \sqrt{x-1}$

16. $\sqrt{x} + 2 = \sqrt{x+4}$ **17.** $\sqrt{2x+1} = 1 + \sqrt{x}$ **18.** $\sqrt{2x-1} - \sqrt{x-1} = 1$