8.5. Dividing radical expressions. Rationalization. Professor Luis Fernández

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Dividing radical expressions.

Recall: As we saw in section 8.2, the quotient property for roots is

$$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}} \qquad \text{or} \qquad \frac{\sqrt[n]{x}}{\sqrt[n]{y}} = \sqrt[n]{\frac{x}{y}}.$$

So to simplify the radical of a fraction,

- Simplify the fraction in the radicand, if possible.
- Use the Quotient Property to rewrite the radical as the quotient of two radicals.
- Simplify the radicals in the numerator and the denominator.

Example: Simplify $\sqrt{\frac{45}{40}}$.

Let us first simplify the fraction: divide numerator and denominator by 5 to get $\frac{45}{40} = \frac{\overset{9}{\cancel{45}}}{\underset{8}{\cancel{40}}} = \frac{9}{8}$.

Therefore $\sqrt{\frac{45}{40}} = \sqrt{\frac{9}{8}} = \frac{\sqrt{9}}{\sqrt{4}\sqrt{2}} = \frac{3}{2\sqrt{2}}$.

Example: Simplify $\sqrt{\frac{14x^2}{36x^3}}$.

Let us first simplify the fraction: $\frac{14x^7}{36x^3} = \frac{\overset{7}{\cancel{14}}\, x^{2-3}}{\underset{18}{\cancel{36}}} = \frac{7x^{-1}}{18} = \frac{7}{18x}$. Therefore $\sqrt{\frac{7}{18x}} = \frac{\sqrt{7}}{\sqrt{9}\sqrt{2x}} = \frac{\sqrt{7}}{3\sqrt{2x}}$.

If instead of having a fraction inside the radical you have a fraction of radicals *of the same index*, one method is to first rewrite it under the same radical sign, simplify the fraction, and then split them again into radicals.

If the radicals have different index, it is easier to use rational exponents as in section 8.3.

Example: Simplify $\frac{\sqrt[3]{40x^4}}{\sqrt[3]{270x^6}}$.

Write it under a single radical sign first: $\frac{\sqrt[3]{40x^4}}{\sqrt[3]{270x^6}} = \sqrt[3]{\frac{40x^4}{270x^6}}$.

Then simplify the fraction (divide numerator and denominator by 5): $\frac{40x^4}{270x^6} = \frac{\overset{8}{\cancel{40}}\, x^{4-6}}{\underset{54}{\cancel{270}}} = \frac{8x^{-2}}{54} = \frac{8}{54x^2}$.

Therefore $\frac{\sqrt[3]{40x^4}}{\sqrt[3]{270x^6}} = \sqrt[3]{\frac{8}{54x^2}} = \frac{\sqrt[3]{8}}{\sqrt[3]{54x^2}} = \frac{2}{\sqrt[3]{27}\sqrt[3]{2x^2}} = \frac{2}{3\sqrt[3]{2x^2}}$.

Exercises:

1. $\sqrt{\frac{50}{80}}$ **2.** $\sqrt[3]{\frac{16}{54}}$ **3.** $\sqrt[3]{\frac{12}{27}}$ **4.** $\sqrt[4]{\frac{32}{160}}$

5. $\sqrt{\frac{x^9}{x^6}}$ **6.** $\sqrt[3]{\frac{y^2}{y^7}}$ **7.** $\frac{\sqrt[5]{x^{12}}}{\sqrt[5]{x^7}}$ **8.** $\frac{\sqrt[7]{x^2}}{\sqrt[7]{x^{12}}}$

9. $\sqrt{\frac{50x^7}{80x^3}}$ **10.** $\frac{\sqrt[3]{40x^5}}{\sqrt[3]{15x^3}}$ **11.** $\frac{\sqrt[4]{5x^5}}{\sqrt[3]{32x^9}}$ **12.** $\frac{\sqrt[3]{40xy}}{\sqrt[3]{5xy^7}}$

When dividing radicals, we have learned to first simplify the fraction and then simplify the radicals. But we could do it the other way around: first simplify the radicals in the numerator and denominator and then simplify the fraction (with radicals) that you obtain. Which of the two methods is better depends on the exercise.

For now, let us do the last example again simplifying radicals first and then simplifying the fraction:

Example: Simplify $\dfrac{\sqrt[3]{40x^4}}{\sqrt[3]{270x^6}}$.

Notice that $40 = 8 \cdot 5$ and $270 = 27 \cdot 2 \cdot 5$. Then

$$\frac{\sqrt[3]{40x^4}}{\sqrt[3]{270x^6}} = \frac{\sqrt[3]{8x^3}\,\sqrt[3]{5}\,\sqrt[3]{x}}{\sqrt[3]{27x^6}\,\sqrt[3]{2}\,\sqrt[3]{5}} = \frac{2\cancelto{1}{x}\;\cancel{\sqrt[3]{5}}\,\sqrt[3]{x}}{3\cancelto{x}{x^2}\;\sqrt[3]{2}\,\cancel{\sqrt[3]{5}}} = \frac{2\,\sqrt[3]{x}}{3x\,\sqrt[3]{2}}\ .$$

It may seem that we got two different answers for the same exercise. Which one is correct?

In fact, the two answers are actually equal, and they are both correct, but to see that they are we need to learn about *rationalization*:

Rationalizing denominators that have only one term

The goal of rationalizing is to rewrite the expression *without any radicals in the denominator*. Every expression can be written this way in a unique way, so it gives the possibility of comparing rational expressions.

The idea is to multiply numerator and denominator of the fraction by the same quantity in such a way that the radicals in the denominator can be simplified.

Example: Rationalize $\dfrac{5}{\sqrt{3}}$.

Multiply numerator and denominator by $\sqrt{3}$. Then simplify: $\dfrac{5}{\sqrt{3}} = \dfrac{5\sqrt{3}}{\sqrt{3}\sqrt{3}} = \dfrac{5\sqrt{3}}{\sqrt{9}} = \dfrac{5\sqrt{3}}{3}$.

That's it! As you can see, we have rewritten the original expression without radicals in the denominator.

Example: Rationalize $\dfrac{\sqrt{10}}{2\sqrt{5}}$.

Multiply numerator and denominator by $\sqrt{5}$. Then simplify: $\dfrac{\sqrt{10}}{2\sqrt{5}} = \dfrac{\sqrt{10}\sqrt{5}}{2\sqrt{5}\sqrt{5}} = \dfrac{\sqrt{25}\sqrt{2}}{2 \cdot 5} = \dfrac{\cancelto{1}{5}\,\sqrt{2}}{\cancelto{2}{10}} = \dfrac{\sqrt{2}}{2}$.

Thus, when the denominator has a square root, in order to rationalize the expression we only have to multiply numerator and denominator by that square root and simplify. **Note:** this only works for square root; for higher index we will see it below.

Example: Rationalize $\dfrac{\sqrt{3}}{5\sqrt{7x}}$.

Multiply numerator and denominator by $\sqrt{7x}$. Then simplify: $\dfrac{\sqrt{3}}{5\sqrt{7x}} = \dfrac{\sqrt{3}\sqrt{7x}}{2\sqrt{7x}\sqrt{7x}} = \dfrac{\sqrt{21x}}{2 \cdot 7x} = \dfrac{\sqrt{21x}}{14x}$.

Exercises: Rationalize. Simplify when possible.

13. $\dfrac{3}{\sqrt{3}}$ **14.** $\dfrac{6}{\sqrt{2}}$ **15.** $\dfrac{4\sqrt{3}}{3\sqrt{5}}$ **16.** $\dfrac{1}{\sqrt{2}}$

17. $\dfrac{3}{2\sqrt{3}}$ **18.** $\dfrac{6\sqrt{x}}{\sqrt{5x}}$ **19.** $\dfrac{5\sqrt{2}}{\sqrt{10}}$ **20.** $\dfrac{6}{\sqrt{2x^3}}$

So far we have only rationalized expressions that have square roots in the denominator. What if the index is not 2? Let us do an example:

Example: Rationalize $\dfrac{2}{\sqrt[3]{7}}$.

Multiply numerator and denominator by $\sqrt[3]{7^2}$. Then simplify: $\dfrac{2}{\sqrt[3]{7}} = \dfrac{2\sqrt[3]{7^2}}{\sqrt[3]{7}\sqrt[3]{7^2}} = \dfrac{2\sqrt[3]{7^2}}{\sqrt[3]{7^3}} = \dfrac{2\sqrt[3]{7^2}}{7}$.

Notice that the point is to multiply numerator and denominator by something which lets us simplify the radical in the denominator. In the last example we multiplied by $\sqrt[3]{7^2}$ because $\sqrt[3]{7} \cdot \sqrt[3]{7^2} = \sqrt[3]{7^3} = 7$.

Example: Rationalize $\dfrac{2}{\sqrt[4]{5}}$.

Multiply numerator and denominator by $\sqrt[4]{5^3}$. Then simplify: $\dfrac{2}{\sqrt[4]{5}} = \dfrac{2\sqrt[4]{5^3}}{\sqrt[4]{5}\sqrt[4]{5^3}} = \dfrac{2\sqrt[4]{5^3}}{\sqrt[4]{5^4}} = \dfrac{2\sqrt[4]{5^3}}{5}$.

Thus, *multiply numerator and denominator by whatever you need to make the radicand in the denominator a multiple of the index.* For example, if we had $\sqrt[5]{x^7}$ in the denominator, we would multiply by $\sqrt[5]{x^3}$, because $x^7 \cdot x^3 = x^{10}$, and 10 is a multiple of 5, so $\sqrt[5]{x^7}\sqrt[5]{x^3} = \sqrt[5]{x^{10}} = x^2$, which has no radicals.

Exercises: Rationalize. Simplify when possible.

21. $\dfrac{1}{\sqrt[3]{5}}$ **22.** $\dfrac{6}{\sqrt[4]{2}}$ **23.** $\dfrac{4\sqrt[4]{3}}{3\sqrt[4]{5}}$ **24.** $\dfrac{1}{\sqrt[5]{2}}$

25. $\dfrac{1}{3\sqrt[5]{7^3}}$ **26.** $\dfrac{6\sqrt[3]{x}}{\sqrt[3]{25x^4}}$ **27.** $\dfrac{5\sqrt[4]{2}}{\sqrt[4]{8}}$ **28.** $\dfrac{6}{\sqrt[4]{2x^3}}$

Rationalizing denominators that have two terms

Here we learn how to rationalize when the denominator has only two terms and these terms involve only square roots, for example,

$$\frac{3}{\sqrt{5}+3} \quad \text{or} \quad \frac{4}{\sqrt{7}-\sqrt{6}}.$$

Again, the idea is to multiply by some appropriate quantity so that the radicals in the denominator disappear.

Recall from the previous section that, using the formula $(a+b)(a-b) = a^2 - b^2$, we have, for example

$$(\sqrt{11}+\sqrt{7})(\sqrt{11}-\sqrt{7}) = (\sqrt{11})^2 - (\sqrt{7})^2 = 11 - 7 = 4.$$

Thus, by multiplying by the same binomial but changing the sign of one of the terms, we get rid of all the radicals.

Given a binomial, the binomial obtained by changing the sign of one of the terms is called the *conjugate*. For example,

- The conjugate of $\sqrt{5}+3$ is $\sqrt{5}-3$.
- The conjugate of $\sqrt{2}-\sqrt{7}$ is $\sqrt{2}+\sqrt{7}$.
- The conjugate of $8+7\sqrt{51}$ is $8-7\sqrt{51}$.

Thus, **to rationalize a radical expression that has a binomial with square roots in the denominator one only needs to multiply numerator and denominator by the conjugate of the denominator**.

Example: Rationalize $\dfrac{5}{4+\sqrt{3}}$.

The conjugate of $4+\sqrt{3}$ is $4-\sqrt{3}$. Multiplying numerator and denominator by $4-\sqrt{3}$ we get

$$\frac{5}{4+\sqrt{3}} = \frac{5(4-\sqrt{3})}{(4+\sqrt{3})(4-\sqrt{3})} = \frac{5(4-\sqrt{3})}{4^2-(\sqrt{3})^2} = \frac{5(4-\sqrt{3})}{16-3} = \frac{5(4-\sqrt{3})}{13}.$$

Example: Rationalize $\dfrac{5}{\sqrt{11}-\sqrt{6}}$.

The conjugate of $(\sqrt{11}-\sqrt{6})$ is $(\sqrt{11}+\sqrt{6})$. Multiplying numerator and denominator by $(\sqrt{11}+\sqrt{6})$ we get

$$\frac{5}{\sqrt{11}-\sqrt{6}} = \frac{5(\sqrt{11}+\sqrt{6})}{(\sqrt{11}-\sqrt{6})(\sqrt{11}+\sqrt{6})} = \frac{5(\sqrt{11}+\sqrt{6})}{(\sqrt{11})^2-(\sqrt{6})^2} = \frac{5(\sqrt{11}+\sqrt{6})}{11-6} = \frac{\cancel{5}(\sqrt{11}+\sqrt{6})}{\cancel{5}} = \sqrt{11}+\sqrt{6}.$$

Exercises: Rationalize. Simplify when possible.

29. $\dfrac{8}{1-\sqrt{6}}$

30. $\dfrac{6}{2-\sqrt{6}}$

31. $\dfrac{5}{4-\sqrt{11}}$

32. $\dfrac{\sqrt{7}}{\sqrt{x}-\sqrt{6}}$

33. $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$

34. $\dfrac{\sqrt{x}-\sqrt{2}}{\sqrt{x}-\sqrt{2}}$

Simplified form of a radical expression

All the methods that we have learned in Chapter 8 allow us to simplify a radical expression completely. A radical expression is considered simplified if

- No factors in the radicand have perfect powers of the index.
- There are no fractions in the radicand.
- There are no radicals in the denominator.

Each radical expression has one and only one simplified form. Writing expressions in simplified form is useful in order to compare expressions.