8.2. Simplify radical expressions Professor Luis Fernández

Radical expressions.

Recall: A radical expression is a mathematical expression involving roots (aka *radicals*). For example,

$$\sqrt{12x^3y^4} \qquad \sqrt{\frac{50x^3y^5}{28z^2}} \qquad \sqrt[3]{24} - 5\sqrt[3]{8},$$

are all radical expressions.

Same as we simplified polynomial expressions or rational expressions, we will now learn how to simplify radical expressions.

Simplifying simple radical expressions

Recall: The product property of exponents says

$$(xy)^n = x^n y^n.$$

A from this property one can see that there is a similar property for roots:

$$\boxed{\sqrt[n]{x \cdot y} = \sqrt[n]{x} \cdot \sqrt[n]{y}.}$$

This property gives the first way for simplifying roots. For example,

$$\sqrt{20} = \sqrt{4 \cdot 5} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5}.$$

Thus, we can rewrite $\sqrt{20}$ in a simpler form as $2\sqrt{5}$.

Example: Simplify $\sqrt{50}$.

50 can be written as $25 \cdot 2$, and $25 = 5^2$. Therefore, $\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$.

Example: Simplify $\sqrt{40}$.

40 can be written as $4 \cdot 10$, and $4 = 2^2$. Therefore, $\sqrt{40} = \sqrt{4 \cdot 10} = \sqrt{4} \cdot \sqrt{10} = 2\sqrt{10}$.

Thus, to simplify the square root of a number,

- Find the greatest factor of the radicand that is a perfect square (that is, the square of a whole number).
- Rewrite the radicand using that factor, split the square root into the product of two square roots, and simplify.

Sometimes it is easier to do it in steps. Just find a factor of the radicand that is a perfect square and do the procedure above. Then, for the remaining radical, repeat the procedure until the remaining radicand has no factors that are perfect squares.

Example: Simplify $\sqrt{80}$.

$80 = 4 \cdot 20$, so $\sqrt{80} = \sqrt{4} \cdot \sqrt{20} = 2\sqrt{20}$.

Now, $20 = 4 \cdot 5$, so $2\sqrt{20} = 2\sqrt{4} \cdot \sqrt{5} = 2 \cdot 2\sqrt{5} = 4\sqrt{5}$. Therefore, $\sqrt{80} = 4\sqrt{5}$.

Of course, it woul have been faster if we had written $80 = 16 \cdot 5$, but that is harder unless you see it immediately. It is easier to do simple steps.

Exercises: Simplify the following radicals:

1. $\sqrt{8}$ **2.** $\sqrt{12}$ **3.** $\sqrt{18}$ **4.** $\sqrt{20}$

5. $\sqrt{24}$ **6.** $\sqrt{26}$ **7.** $\sqrt{27}$ **8.** $\sqrt{28}$

9. $\sqrt{30}$ **10.** $\sqrt{32}$ **11.** $\sqrt{42}$ **12.** $\sqrt{44}$

13. $\sqrt{45}$ **14.** $\sqrt{63}$ **15.** $\sqrt{75}$ **16.** $\sqrt{96}$

The same procedure can be used to simplify roots of any index:

Example: Simplify $\sqrt[4]{48}$.

48 can be written as $16 \cdot 4$, and $16 = 2^4$, so that $\sqrt[4]{16} = 2$. Therefore, $\sqrt[4]{48} = \sqrt[4]{16 \cdot 3} = \sqrt[4]{16} \cdot \sqrt[4]{3} = 2\sqrt[4]{3}$.

So, in general, to simplify $\sqrt[n]{x}$,

- Write x as $a^n \cdot b$ (if possible). If it is not possible, then it cannot be simplified further.
- Then $\sqrt[n]{x} = \sqrt[n]{a^n}\,\sqrt[n]{b} = a\,\sqrt[n]{b}$.

Exercises: Simplify the following radicals:

17. $\sqrt[3]{24}$ **18.** $\sqrt[3]{250}$ **19.** $\sqrt[3]{54}$ **20.** $\sqrt[3]{75}$

21. $\sqrt[5]{64}$ **22.** $\sqrt[4]{48}$ **23.** $\sqrt[3]{250}$ **24.** $\sqrt[3]{2000}$

When the expressions have variables, it is actually easier. For example, $\sqrt{x^3} = \sqrt{x^2 \cdot x} = \sqrt{x^2} \cdot \sqrt{x} = |x|\sqrt{x}$ (do not forget the absolute value when the index is even.

Example: Simplify $\sqrt{x^{11}}$.

We can write x^{11} as $x^{10} \cdot x$. Therefore $\sqrt{x^{11}} = \sqrt{x^{10}} \cdot \sqrt{x} = |x^5|\sqrt{x}$.

Example: Simplify $\sqrt[3]{x^{11}}$.

We can write x^{11} as $x^9 \cdot x^2$. Therefore $\sqrt[3]{x^{11}} = \sqrt[3]{x^9} \cdot \sqrt[3]{x^2} = x^3\,\sqrt[3]{x^2}$.

If there are several variables, or variables and numbers, do each one separately and then multiply the results.

Example: Simplify $\sqrt{x^5y^7}$.

Since $5 = 4 + 1$, $\sqrt{x^5} = \sqrt{x^4 \cdot x} = \sqrt{x^4} \cdot \sqrt{x} = \sqrt{(x^2)^2} \cdot \sqrt{x} = |x^2|\sqrt{x}$.

Also, since $7 = 6 + 1$, $\sqrt{y^7} = \sqrt{y^6} \cdot \sqrt{y} = \sqrt{(y^3)^2} \cdot \sqrt{y} = |y^3|\sqrt{y}$.

Therefore, $\sqrt{x^5y^7} = \sqrt{x^5}\sqrt{y^7} = |x^2||y^3|\sqrt{xy}$.

Example: Simplify $\sqrt[3]{24x^5y^7}$.

Do $\sqrt[3]{24}$, $\sqrt[3]{x^5}$, and $\sqrt[3]{y^7}$ separately: $\sqrt[3]{24} = \sqrt[3]{8 \cdot 3} = \sqrt[3]{8} \cdot \sqrt[3]{3} = 2\sqrt[3]{3}$.

$\sqrt[3]{x^5} = \sqrt[3]{x^3 \cdot x^2} = \sqrt[3]{x^3} \cdot \sqrt[3]{x^2} = x\sqrt[3]{x^2}$. $\sqrt[3]{y^7} = \sqrt[3]{y^6 \cdot y} = \sqrt[3]{y^6} \cdot \sqrt[3]{y} = y^2\,\sqrt[3]{y}$

Putting it all together, $\sqrt[3]{24x^5y^7} = 2xy^2\,\sqrt[3]{3x^2y}$.

Exercises: Simplify the following radicals:

25. $\sqrt{x^9}$ **26.** $\sqrt{y^5}$ **27.** $\sqrt{z^{21}}$ **28.** $\sqrt{18x^5}$

29. $\sqrt{12x^{13}}$ **30.** $\sqrt{50x^7y^4}$ **31.** $\sqrt{15x^3y^9z^5}$ **32.** $\sqrt{20x^5y^9}$

33. $\sqrt[3]{x^4}$ **34.** $\sqrt[3]{y^7}$ **35.** $\sqrt[3]{z^5}$ **36.** $\sqrt[3]{x^{25}}$

37. $\sqrt[5]{x^{13}}$ **38.** $\sqrt[4]{32x^9}$ **39.** $\sqrt[3]{24x^5y^{10}}$ **40.** $\sqrt[3]{4000x^{91}}$

Simplifying radicals of fractions

Recall: The quotient rule for exponents says

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} \qquad \text{or} \qquad \frac{x^n}{y^n} = \left(\frac{x}{y}\right)^n.$$

The equivalent rule for roots is

$$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}} \qquad \text{or} \qquad \frac{\sqrt[n]{x}}{\sqrt[n]{y}} = \sqrt[n]{\frac{x}{y}}.$$

Therefore, to simplify the n^{th} root of a fraction we can just find the n^{th} root of the numerator and the n^{th} root of the denominator.

Example: Simplify $\sqrt{\frac{45}{20}}$.

Let us first simplify the fraction: $\frac{45}{20} = \frac{\cancel{45}^{\,9}}{\cancel{20}_{\,4}} = \frac{9}{4}$. Therefore $\sqrt{\frac{45}{20}} = \sqrt{\frac{9}{4}} = \frac{\sqrt{9}}{\sqrt{4}} = \frac{3}{2}$.

Example: Simplify $\sqrt{\frac{14x^7}{72x^3}}$.

Let us first simplify the fraction: $\frac{14x^7}{72x^3} = \frac{\cancel{14}^{\,7}\, x^{7-3}}{\cancel{72}_{\,36}} = \frac{7x^4}{36}$. Therefore $\sqrt{\frac{14x^7}{72x^3}} = \frac{\sqrt{7}\sqrt{x^4}}{\sqrt{36}} = \frac{\sqrt{7}x^2}{6}$.

Example: Simplify $\sqrt[3]{\frac{16x^8}{54x^5}}$.

Let us first simplify the fraction: $\frac{16x^8}{54x^5} = \frac{\cancel{16}^{\,8}\, x^{8-5}}{\cancel{54}_{\,27}} = \frac{8x^3}{27}$. Therefore $\sqrt[3]{\frac{16x^8}{54x^5}} = \frac{\sqrt[3]{8}\ \sqrt[3]{x^3}}{\sqrt[3]{27}} = \frac{2x}{3}$.

So you can see that to simplify the radical of a fraction,

- Simplify the fraction in the radicand, if possible.
- Use the Quotient Property to rewrite the radical as the quotient of two radicals.
- Simplify the radicals in the numerator and the denominator.

Exercises: Simplify the following radicals:

41. $\sqrt{\frac{45}{80}}$ **42.** $\sqrt[3]{\frac{16}{27}}$ **43.** $\sqrt[3]{\frac{24}{81}}$ **44.** $\sqrt[4]{\frac{32}{162}}$

45. $\sqrt{\frac{x^{10}}{x^6}}$ **46.** $\sqrt[3]{\frac{y^{11}}{y^2}}$ **47.** $\sqrt[5]{\frac{x^{12}}{x^7}}$ **48.** $\sqrt[6]{\frac{y^{30}}{y^{12}}}$

49. $\sqrt{\frac{50x^7}{98x^3}}$ **50.** $\sqrt[3]{\frac{15x^5}{40x^3}}$ **51.** $\sqrt{\frac{54x^5}{450x}}$ **52.** $\sqrt[4]{\frac{10x^9}{32x}}$