7.1. Multiply and divide rational expressions Professor Luis Fernández

Rational expressions. Simplifying rational expressions.

A *rational expression* is a mathematical expression that can be written in the form $\frac{p}{q}$, where p and q are polynomials. For example

$$\frac{x^2+1}{x+5} \qquad \frac{x^5-6x+1}{x^3+5} \qquad \frac{1}{x^3+5} \qquad \frac{(x+2)(x-1)}{(x-3)(x-2)} \qquad 3x+1\left(=\frac{3x+1}{1}\right).$$

Rational expressions are just fractions of polynomials, so they have similar properties as fractions of numbers. In particular, they can be simplified: Recall that a fraction is in *simplified form* if the numerator and the denominator have no common factors. For example,

$\frac{12}{16}$ is not simplified because 4 is a common factor of 12 and 16, whereas

$\frac{5}{6}$ is simplified because 5 and 6 have no common factors (besides 1, of course)

Recall: to simplify a fraction one has to divide the numerator and the denominator by the greatest common factor. For example, dividing numerator and denominator by 4 we get $\frac{12}{16}=\frac{3}{4}$.

Another way of doing this is to rewrite the original fraction as a product, and then cancel out those terms that appear both in the numerator and the denominator:

$$\frac{12}{16}=\frac{4\cdot 3}{4\cdot 4}=\frac{\cancel{4}\cdot 3}{\cancel{4}\cdot 4}=\frac{3}{4}.$$

Exactly the same method is used to simplify rational expressions. For example, to simplify $\frac{(3x+1)(x-3)}{(3x+1)(x-2)}$, do

$$\frac{(3x+1)(x-3)}{(3x+1)(x-2)}=\frac{\cancel{(3x+1)}(x-3)}{\cancel{(3x+1)}(x-2)}=\frac{x-3}{x-2}.$$

Note that in order to do this we first need to factor the numerator and the denominator. For example,

Example: Simplify (if possible) the rational expression $\frac{x^2-5x+6}{x^2-4}$.

First let us factor the numerator and the denominator. x^2-5x+6 is factored as $x^2-5x+6=(x-3)(x-2)$, and x^2-4 is a difference of squares: $x^2-4=(x+2)(x-2)$. Therefore the rational expression can be written as

$$\frac{x^2-5x+6}{x^2-4}=\frac{(x-3)(x-2)}{(x+2)(x-2)}=\frac{(x-3)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}=\frac{x-3}{x+2}.$$

Example: Simplify (if possible) the rational expression $\frac{6x^4-18x^3+12x^2}{3x^5-12x^3}$.

First let us factor the numerator. In $6x^4-18x^3+12x^2$ we notice that $6x^2$ is a common factor of all the terms, so we factor it out: $6x^4-18x^3+12x^2=6x^2(x^2-3x+2)$. Then notice the the second factor, that is, x^2-3x+2, can be factored as $x^2-3x+2=(x-1)(x-2)$. Therefore the numerator is completely factored as

$$6x^4-18x^3+12x^2=6x^2(x-1)(x-2).$$

For the denominator, $3x^3$ is a common factor of all the terms of $3x^5-12x^3$, and we get $3x^5-12x^3=3x^3(x^2-4)$. Then notice that the second factor, that is, x^2-4, is a difference of squares: $x^2-4=(x+2)(x-2)$. Therefore we have

$$\frac{6x^4-18x^3+12x^2}{3x^5-12x^3}=\frac{6x^2(x-1)(x-2)}{3x^3(x+2)(x-2)}=\frac{2\cdot 3\cdot x^2(x-1)(x-2)}{3\cdot x\cdot x^2(x+2)(x-2)}=\frac{2\cdot\cancel{3}\cdot\cancel{x^2}(x-1)\cancel{(x-2)}}{\cancel{3}\cdot x\cdot\cancel{x^2}(x+2)\cancel{(x-2)}}=\frac{2(x-1)}{x(x+2)}.$$

Note that some rational expressions cannot be simplified. For example, $\dfrac{x^2+4x+4}{x^2+4x+3} = \dfrac{(x+2)(x+2)}{(x+3)(x+1)}$, which has no common factors between the numerator and the denominator. In this case, just leave the numerator and denominator factored and you are done.

Note also: sometimes the numerator and denominator have factors that are equal except for the sign. For example,

$$\frac{(x+3)(x-2)}{(x+4)(2-x)}.$$

Note that $(x-2)$ and $(2-x)$ is the same factor except for the sign. In other words, we can write $(2-x) = -1\cdot(x-2)$. Then we can simplify:

$$\frac{(x+3)(x-2)}{(x+4)(2-x)} = \frac{(x+3)(x-2)}{(x+4)\cdot(-1)\cdot(x-2)} = \frac{(x+3)\cancel{(x-2)}}{(x+4)\cdot(-1)\cdot\cancel{(x-2)}} = \frac{x+3}{(-1)(x+4)} = -\frac{x+3}{x+4}.$$

Summarizing, to simplify a rational expression,

- Factor the numerator and the denominator completely.
- Cancel common factors.

Practice exercises: Simplify, if possible, the following rational expressions.

1. $\dfrac{(x+2)(x-3)}{(x+2)(x-5)}$

2. $\dfrac{(x+4)(x^2+1))}{(x^2+1)(x-5)}$

3. $\dfrac{16(x+5)^2(x-3)}{12(x+2)^2(x+5)}$

4. $\dfrac{x^3-2x^2-25x+50}{x^2-5x}$

5. $\dfrac{x^2+8x+15}{x^2-9}x$

6. $\dfrac{x^2+2x-15}{x^2+6x+5}$

7. $\dfrac{49-x^2}{x^2+8x+7}$

8. $\dfrac{x-7}{7-x}$

9. $\dfrac{x^2+7x+12}{x^2+3x+2}$

10. $\dfrac{-5x^2-10x}{-10x^2+30x+100}$

Multiplication of rational expressions

To multiply rational expressions proceed exactly as with fractions of numbers: multiply the numerators, multiply the denominators, and then simplify.

Example: Multiply $\dfrac{x+3}{x-2}\cdot\dfrac{x-2}{x+5}$.

$$\frac{x+3}{x-2}\cdot\frac{x-2}{x+5}=\frac{(x+3)(x-2)}{(x-2)(x+5)}=\frac{(x+3)\cancel{(x-2)}}{\cancel{(x-2)}(x+5)}=\frac{x+3}{x+5}.$$

Example: Multiply $\dfrac{x^2+4x+4}{x^2-9}\cdot\dfrac{x^2+4x+3}{x^2+x-2}$.

Start by factoring all the polynomials:

$$x^2+4x+4=(x+2)^2 \qquad x^2-9=(x+3)(x-3)$$
$$x^2+4x+3=(x+3)(x+1) \qquad x^2+x-2=(x+2)(x-1).$$

Therefore we can write

$$\begin{aligned}\frac{x^2+4x+4}{x^2-9}\cdot\frac{x^2+4x+3}{x^2+x-2}&=\frac{(x+2)(x+2)}{(x+3)(x-3)}\cdot\frac{(x+3)(x+1)}{(x+2)(x-1)}\\&=\frac{(x+2)(x+2)(x+3)(x+1)}{(x+3)(x-3)(x+2)(x-1)}=\frac{\cancel{(x+2)}(x+2)\cancel{(x+3)}(x+1)}{\cancel{(x+3)}(x-3)\cancel{(x+2)}(x-1)}=\frac{(x+2)(x+1)}{(x-3)(x-1)}.\end{aligned}$$

Practice exercises: Multiply and simplify.

11. $\dfrac{x+2}{x-3}\cdot\dfrac{x+1}{x+2}$

12. $\dfrac{(x+4)(x+3)}{(x-2)(x-5)}\cdot\dfrac{(x+4)(x-5)}{(x+3)(x+5)}$

13. $\dfrac{x^2+5x+6}{x^2+2x-3}\cdot\dfrac{x^2+7x+12}{x+2}$

14. $\dfrac{x^2+3x}{x^2-3x-4}\cdot\dfrac{(x-4)(x-5)}{x^2}$

15. $\dfrac{72x-12x^2}{8x+32}\cdot\dfrac{x^2+10x+24}{36x^2-1}$

16. $\dfrac{3x^2+15x}{x^2+10x+25}\cdot\dfrac{1}{6x^2+30x}$

Division of rational expressions

To multiply rational expressions, also proceed exactly as with fractions of numbers: change the fraction after the division sign to its reciprocal, change the division sign to a multiplication sign, and then multiply as before.

Example: Divide: $\dfrac{x+3}{x-1}\div\dfrac{x+3}{x+5}$.

$$\frac{x+3}{x-1}\div\frac{x+3}{x+5}=\frac{x+3}{x-1}\cdot\frac{x+5}{x+3}=\frac{(x+3)(x+5)}{(x-1)(x+3)}=\frac{\cancel{(x+3)}(x+5)}{(x-1)\cancel{(x+3)}}=\frac{x+5}{x-1}.$$

Example: Divide $\dfrac{x^2-9}{x^2+4x+4} \div \dfrac{x^2+4x+3}{x^2+x-2}$.

Start by factoring all the polynomials:

$$x^2+4x+4 = (x+2)^2 \qquad x^2-9=(x+3)(x-3)$$
$$x^2+4x+3=(x+3)(x+1) \qquad x^2+x-2=(x+2)(x-1).$$

Therefore we can write

$$\frac{x^2-9}{x^2+4x+4} \div \frac{x^2+4x+3}{x^2+x-2} = \frac{x^2-9}{x^2+4x+4} \cdot \frac{x^2+x-2}{x^2+4x+3} = \frac{(x+3)(x-3)}{(x+2)(x+2)} \cdot \frac{(x+2)(x-1)}{(x+3)(x+1)}$$
$$= \frac{(x+3)(x-3)(x+2)(x-1)}{(x+2)(x+2)(x+3)(x+1)} = \frac{\cancel{(x+3)}(x-3)\cancel{(x+2)}(x-1)}{\cancel{(x+2)}(x+2)\cancel{(x+3)}(x+1)} = \frac{(x-3)(x-1)}{(x+2)(x+1)}.$$

Note: one can write division using the symbol "÷" or using a fraction sign, as in $\dfrac{8}{4} = 8 \div 4$. So another way to write division of rational expressions is, for example,

$$\frac{\frac{4x}{x-3}}{\frac{6x^2}{x+2}}.$$

If you prefer, you can rewrite it using the symbol "÷" and then proceed as before. In this case, it would be $\dfrac{4x}{x-3} \div \dfrac{6x^2}{x+2}$. You can also jump directly and multiply the numerator of the big fraction by the reciprocal of the denominator of the big fraction.

Practice exercises: Divide and simplify.

17. $\dfrac{x+2}{x-1} \div \dfrac{x+2}{x+5}$

18. $\dfrac{(x-3)(x+3)}{(x-7)(x-6)} \div \dfrac{(x-3)(x-5)}{(x-7)(x+5)}$

19. $\dfrac{x^2+5x+6}{x^2+7x+12} \div \dfrac{x+2}{x^2+2x-3}$

20. $\dfrac{\frac{x^2-4x}{x^2-3x-10}}{\frac{(x-4)}{x^2(x-5)}}$

21. $\dfrac{\frac{12x-72x^2}{8x-32}}{\frac{x^2-36}{x^2-10x+24}}$

22. $\dfrac{\frac{1}{x^2-10x+25}}{\frac{1}{5x^3-25x^2}}$