6.5. Polynomial equations Professor Luis Fernández

Polynomial equations

A *polynomial equation* is an equation that involves a polynomial. For example, $x^2 + 3x - 4 = 0$ is a polynomial equation.

The *degree* of a polynomial equation is the degree of the polynomial involved. For example, $x^2 + 3x - 4 = 0$ is an equation of degree 2.

- Equations of degree 2 are also called *quadratic*.
- Equations of degree 3 are also called *cubic*. Thus, $x^2 + x = 5$ is a quadratic equation and $x^3 + 5x - 5 = x^2$ is a cubic equation, whereas $x^4 + 5x^3 - 4x = 7$ is an equation of degree 4.

In this course we will learn how to solve quadratic equations. The first method we will learn involves *factoring*.

The idea is a property of numbers:

$$\text{If } a \cdot b = 0, \text{ then either } a = 0 \text{ or } b = 0 \text{ (or both).}$$

Thus, if we have an equation where the left hand side is a factored polynomial and the right hand side is 0, to find the solution we only have to set each factor equal to 0 and solve the simple equations we get. For example

<u>Example</u>: Solve $x^2 - 5x + 6 = 0$.

Let us factor the polynomial: we have $x^2 - 5x + 6 = (x - 2)(x - 3)$. Thus we can rewrite the equation as

$$(x - 2)(x - 3) = 0.$$

But then this implies that either $x - 2 = 0$ or $x - 3 = 0$. Solving these two equations gives $x = 2$ or $x = 3$. Thus, the solutions of $x^2 - 5x + 6 = 0$ are $x = 2$ and $x = 3$.

One can check that these are solutions: $3^2 - 5 \cdot 3 + 6 = 9 - 15 + 6 = 0$ and $2^2 - 5 \cdot 2 + 6 = 4 - 10 + 6 = 0$.

NOTE: The zero product property only works when the right hand side is 0!! If it is not 0, move all the terms to the left hand side using the usual rules.

<u>Example</u>: Solve $x^2 - 10 = 3x$.

First, we need to write the equation with 0 on the right hand side. To do this, subtract $3x$ from both sides to get $x^2 - 3x - 10 = 0$. Now let us factor the polynomial: $x^2 - 3x - 10 = (x - 5)(x + 2)$. Thus we can rewrite the equation as $(x - 5)(x + 2) = 0$. This means that either $x - 5 = 0$, which gives $x = 5$, or $x + 2 = 0$, which gives $x = -2$. Thus the solutions are $x = 5$ and $x = -2$.

Let us check if $x = 5$ and $x = -2$ are solutions of $x^2 - 10 = 3x$. For $x = 5$, the LHS equals $5^2 - 10 = 25 - 10 = 15$, and the RHS is $3 \cdot 5 = 15$ also, so $x = 5$ is a solution. For $x = -2$, the LHS equals $(-2)^2 - 10 = 4 - 10 = -6$, and the RHS is $3 \cdot (-2) = -6$ also, so $x = -2$ is a solution.

Summarizing, to solve a quadratic equation by factoring,

1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.
2. Factor the quadratic expression on the left.
3. Use the Zero Product Property (that is, set each factor equal to 0).
4. Solve the two linear equations you obtained in 3.
5. Check. Substitute each solution separately into the original equation.

This method also works with equations of any degree as long as the polynomial is factored.

Example: Solve the equation $(x+3)(x-4)(2x-3)(4x+1)=0$.

Since the polynomial is factored, we can use the zero product property immediately: one of the four factors of the polynomial has to be 0. Thus we have

$$x+3=0 \qquad \text{or} \qquad x-4=0 \qquad \text{or} \qquad 2x-3=0 \qquad \text{or} \qquad 4x+1=0,$$

which, solving the equations, gives

$$x=-3 \qquad \text{or} \qquad x=4 \qquad \text{or} \qquad x=\frac{3}{2} \qquad \text{or} \qquad x=-\frac{1}{4}.$$

Thus the equation has four solutions: $-3, 4, 3/2$ and $-1/4$.

Practice exercises: Solve the following equations.

1. $x^2-6x+8=0$

2. $x^2+9x+18=0$

3. $x^2-5x+6=0$

4. $x^2-10=3x$

5. $x^2+4x=12$

6. $x^2=7x+8$

7. $(x+2)(x-3)(3x-5)=0$

8. $x^2=9$

9. $4x^4-20x^3+24x^2=0$

10. $12x^3+12x^2=24x$

11. $100x^2=9$

12. $x^2+15x+36=0$

13. $5x^2+21x+4=0$

14. $8x^2+80x+200=0$

15. $6x^3=24x$

16. $2x^2-27x-45=$

17. $5x^2+4=9x$

18. $4x^2+16x+15=0$