6.1. Greatest common factor and factor by grouping Professor Luis Fernández

Factoring

The goal of this chapter is to learn how to *factor* polynomials. *Factoring* means to write the polynomial as a product of polynomials of smaller degree. For example

$$x^2 + 5x + 6 = (x + 3)(x + 2).$$

You can see factoring as "doing multiplication backwards". We know how to do this with numbers for example,

$$24 = 2^3 \cdot 3.$$

Remember that a number is called *prime* if it does not have any factors other than 1 and itself. For example, the first few prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31,. . .

Practice exercises: Factor the following numbers into prime factors.

1. $12 =$ **2.** $32 =$ **3.** $50 =$ **4.** $18 =$

Factoring out the Greatest Common Factor (GCF) of a polynomial

Recall the distributive property: $a(b + c) = a \cdot b \ + a \cdot c$. To *factor out* the GCF of a polynomial we just use the same property *in reverse*:

$$a \cdot b + a \cdot c = a(b + c).$$

For example, in the polynomial $4x^3 + 10x$, $2x$ is a factor of $4x^3 = 2 \cdot 2 \cdot x \cdot x \cdot x$ and also of $10x = 2 \cdot 5 \cdot x$. Therefore

$$4x^3 + 10x = 2x(2x^2 + 5).$$

You already know how to do this with numbers. For example, the GCF of 30 and 12 is 6: the factors of 30 are 1, 30, 2, 15, 3, 10, 5, 6, and the factors of 12 are 1, 12, 2, 6, 3, 4. We see that the greatest number that is a factor of both is 6.

Practice exercises: Find the GCF of the given numbers.

5. 12 and 16 **6.** 32 and 16 **7.** 25 and 10 **8.** 45 and 60

The GCF of two or more monomials *is the product of the GCF of the coefficients and the GCF of the variables.* Finding the GCF of the variables is easy: it is just the *common* variables raised to the smallest exponent in all the terms.

Example: Find the GCF of $12x^3y^4$ and $15x^2$.

The GCF of 12 and 15 is 3. The common variable is x only (the second monomial does not have a y). The exponents of x are 3 and 2, so the smallest is 2. Therefore the GCF of $12x^3y^4$ and $15x^2$ is $3x^2$.

Example: Find the GCF of $16x^3y^4z^5$ and $12xy^3z^7$.

The GCF of 12 and 16 is 4. All x, y and z are common variables (they appear in both polynomials). The smallest exponent for x is 1, the smallest exponent for y is 3 and the smallest exponent for z is 5. Therefore the GCF of $16x^3y^4z^5$ and $12xy^3z^7$ is $4xy^3z^5$.

Practice exercises: Find the GCF of the given monomials.

9. $12x^3y^2$ and $16y$ **10.** $4x$ and 2 **11.** $6x^2$ and $9x^3$

12. $6x^2$ and $5y^2$ **13.** $12x^2y^5z^3$ and $4xy^2z^4$ **14.** $6(x + 1)^2$ and $4(x + 1)$

Factoring out the greatest common factor in a polynomial

To factor out the greatest common factor,

- Find the GCF of all the terms of the polynomial.
- Rewrite each term as a product using the GCF.
- Use the "reverse" Distributive Property to factor the expression.
- Check by multiplying the factors.

Example: Factor out the GCF of $4x^3 + 6x^2 - 8x$.

The GCF of the numbers is 2, and the GCF of the variables is x. Therefore the GCF is $2x$. We can then write

$$4x^3 + 6x^2 - 8x = 2x \cdot 2x^2 + 2x \cdot 3x - 2x \cdot 4 = 2x(2x^2 + 3x - 4).$$

Example: Factor out the GCF of $6x^2 - 18x$.

The GCF of the numbers is 6, and the GCF of the variables is x. Therefore the GCF is $6x$ and we can write

$$6x^2 - 18x = 6x \cdot x - 6x \cdot 3 = 6x(x - 3).$$

Note that with practice, the second step can be skipped. You only have to remember to divide each term of the polynomial by the GCF and write it inside the parenthesis.

Example: Factor out the GCF of $6x^2y^3 - 9xy^4 + 3y^2$.

The GCF of the numbers is 3, the only common variable is y, with lowest exponent 2, so the GCF of the variables is y^2. Therefore the GCF if $3y^2$. We now divide each term by $3y^2$: $(6x^2y^3)/(3y^2) = 2x^2y$; $(-9xy^4)/(3y^2) = -3xy^2$; $(3y^2)/(3y^2) = 1$. Therefore,

$$6x^2y^3 - 9xy^4 + 3y^2 = 3y^2(2x^2y - 3xy^2 + 1).$$

Practice exercises: Factor out the GCF of the given polynomials.

15. $30x^2 - 5x =$

16. $x^4 - 5x^3 + 6x^2 =$

17. $4x^4y^3 - 8x^2y + 4xy =$

18. $5x^4 - 15x^3 + 25x^2 =$

19. $12x^4y^3 - 18x^3y^2 - 6x^4y^6 =$

20. $x(x + 1) - 5(x + 1) =$

21. $2x^2 - 5 =$

22. $x^2(x^2 + 4) - 4(x^2 + 4) =$

23. $-2x^4 - 10x^3 - 20x^2 =$

24. $x^2(x^2 + 4) - 4(x^2 + 4) =$

Notice that when we factor out we always have two different options for the sign. For example, we can write

$$-2x + 4 = 2(-x + 4) \quad \text{or} \quad -2x + 4 = -2(x - 4).$$

Practice exercises: Factor out the GCF of the given polynomials in two different ways, one with the common factor having positive sign and one with the common factor having negative sign.

25. $30x - 5 =$

26. $-4x^2 + 16 =$

27. $12x^3 - 16x^2 =$

28. $-10x - 30 =$

Factor by grouping

This method *only works for some polynomials with four terms.* It consists of factoring out the GCF of the first two terms and of the last two terms, and then factoring out the GCF of the resulting expressions. For example to factor $xy + 3y + 2x + 6$ we first factor out the y from the first two terms and the 2 from the last two terms to get $y(x+3)+2(x+3)$. Then we see that, luckily (this is very unusual), the two terms have $(x+3)$ as a common factor:

$$xy + 3y + 2x + 6 = y(x + 3) + 2(x + 3) = (x + 3)(y + 2).$$

Again, this method works for very few polynomials, but it is very nice.

Example: Factor $x^3 - 5x^2 + 2x - 10$.

$$\underbrace{x^3 - 5x^2}_{x^2(x-5)} + \underbrace{2x - 10}_{2(x-5)} = x^2(x - 5) + 2(x - 5) = (x^2 + 2)(x - 5).$$

Example: Factor $20x^2 - 16x - 15x + 12$. Notice that, in this case, we can write the first two terms as $20x^2 - 16x = 4x(5x - 4)$ and the last two as $15x + 12 = 3(-5x + 4)$. But then there is a problem: the expressions in parenthesis inside each term are not the same, so they cannot be factored out. But we have another option. We can also write the second term as $15x + 12 = -3(5x - 4)$. Then

$$\underbrace{20x^2 - 16x}_{4x(5x-4)} \underbrace{-15x + 12}_{-3(5x-4)} = x(5x - 4) - 3(5x - 4) = (x - 3)(5x - 4).$$

Practice exercises: Factor out the GCF of the given polynomials.

29. $cd + 6c + 4d + 24 =$

30. $6y^2 + 7y + 24y + 28 =$

31. $x^2 - x + 4x - 4 =$

32. $9p^2 + 12p - 15p - 20 =$

33. $mn - 6m - 4n + 24 =$

34. $x^3 + x^2 + x + 1 =$

35. $5x^3 - 3x^2 + 5x - 3 =$

36. $3x^2 - x + 6xy - 2y =$