4.1. Systems of linear equations Professor Luis Fernández

Systems of two linear equations in two variables

A system of equations is a set of equations that need to be satisfied *simultaneously*. That is, a *solution* of a system of equations is a value for each of the variables that is a solution of **ALL** the equations in the system.

To denote that the equations go together as a system, we write a bracket on the left holding the equations together. For example,

$$\begin{cases} 4x + y = 8 \\ -2x + 3y = 10 \end{cases}$$

is a system of equations.

You can see that, for example, $(0, 8)$ and $(2, 0)$ <u>are solutions of the *first* equation</u>, but they are **not** solutions of the system, because <u>they are not solutions of the second equation</u>.

Likewise, $(-5, 0)$ is a solution of the *second*, equation, but it is not a solution of the first equation, so it is **not** a solution of the system of equations.

However, $(1, 4)$ is a solution of the system, because it is a solution of **both** equations.

<u>Example</u>: Check if $(3, 0)$, $(1, -2)$, and $(0, 1)$ are solutions of the system of equations

$$\begin{cases} x - y = 3 \\ 3x + y = 1 \end{cases}$$

Check $(3, 0)$: substitute it into the first equation: $3 - 0 = 3$, so $(3, 0)$ is a solution of the first equation. Now substitute it into the second equation: $3 \cdot 3 + 0 = 9 \neq 1$, so it is *not* a solution of the second equation, and therefore $(3, 0)$ **is not a solution of the system**.

Check $(1, -2)$: substitute it into the first equation: $1 - (-2) = 1 + 2 = 3$, so $(1, -2)$ is a solution of the first equation. Now substitute it into the second equation: $3 \cdot 1 + (-2) = 3 - 2 = 1$, so it is also a solution of the second equation. Therefore $(1, -2)$ **is a solution of the system**.

Check $(0, 1)$: substitute it into the first equation: $-1 \neq 31$, so $(0, 1)$ is **not** a solution of the first equation. Therefore $(0, 1)$ **is not a solution of the system** (we do not even care whether it is a solution of the second equation).

<u>Practice exercise</u>: Check whether $(1, 4)$, $(5, 2)$, and $(2, 1)$ are solutions of the system

$$\begin{cases} x - 3y = -1 \\ 3x + y = 7 \end{cases}$$

How to solve systems of equations. Elimination method. There are several methods to solve (that is, find the solution of) systems of equations. The elimination methods is perhaps the simplest.

Recall that to solve a single equation we had 4 rules to manipulate them: For systems, we add a fifth rule:

1. You can add **the same** number or expression to **both sides** of the equation.
2. You can subtract **the same** number or expression from **both sides** of the equation.
3. You can multiply **both sides** of the equation by **the same** number or expression.
4. You can divide **both sides** of the equation by **the same** number or expression.

For systems, we add a fifth rule:

5. You can add the two equations, left side with the left side, and right side with the right side.

The idea is to use these rules to *eliminate* one of the variables, and then solve the resulting equation, which will have a single variable. Then, once the solution of one of the variables is found, substitute in one of the other equations, solve it, and find the value of the other variable.

Example: Solve the system $\begin{cases} x - 3y = 1 \\ 2x + 3y = 11 \end{cases}$

If we add the two equations we obtain

$$\begin{array}{rl} x - 3y &= 1 \\ \underline{2x + 3y} &\underline{= 11} \\ 3x \qquad &= 12. \end{array}$$

Then we can solve this last equation to get $x = 4$. This gives the value of x in the solution. To find the value of y, substitute $x = 4$ in either equation (you will get the same result), for example in the first one:

$$4 - 3y = 1 \quad \to \quad -3y = -3 \quad \to \quad y = 1.$$

Therefore the solution of the system above is $(4, 1)$. Please check!

Example: Solve the system $\begin{cases} 3x - 2y = 1 \\ -3x + 3y = 0 \end{cases}$

If we add the two equations we obtain

$$\begin{array}{rl} 3x - 2y &= 1 \\ \underline{-3x + 3y} &\underline{= 0} \\ y &= 1. \end{array}$$

This immediately gives the value of y in the solution ($y = 1$). To find the value of x, substitute $y = 1$ in either equation (you will get the same result), for example in the first one:

$$3x - 2 \cdot 1 = 1 \quad \to \quad 3x = 3 \quad \to \quad x = 1.$$

Therefore the solution of the system above is $(1, 1)$. Please check!

Practice exercises: Solve the following systems

1. $\begin{cases} 2x - y = 7 \\ -2x + 3y = -5 \end{cases}$ **2.** $\begin{cases} 4x - 3y = 5 \\ -x + 3y = 1 \end{cases}$ **3.** $\begin{cases} 5x + y = 6 \\ -5x - 3y = 2 \end{cases}$ **4.** $\begin{cases} 3x + 4y = 13 \\ 2x - 4y = 2 \end{cases}$

The reason why these systems were easy to solve is because the coefficient of one of the variables in the first equation is the opposite of the coefficient of the same variable in the second equation (for example, $2x$ and $-2x$ in exercise 1, or $-3y$ and $3y$ in exercise 2). What can we do if this is not the case?

Then we can use rule 3 above to multiply each equation by a suitable number so that the resulting system can be solved by the method we have learned. For example, consider the system

$$\begin{cases} 2x - 5y = -4 \\ 3x - 4y = 1 \end{cases}$$

If we multiply the first equation by 3 and the second equation by (-2) we obtain

$$\begin{cases} 6x - 15y = -12 \\ -6x + 8y = -2 \end{cases}$$

Then we can add the two equations to eliminate x, as before. The process would be as follows:

$$\begin{cases} 2x - 5y &= -4 \\ 3x - 4y &= 1 \end{cases} \begin{array}{c} \xrightarrow{\times 3} \\ \xrightarrow{\times(-2)} \end{array} \begin{cases} 6x - 15y = -12 \\ -6x + 8y = -2 \end{cases} \xrightarrow{\text{add equations}} -7y = -14 \xrightarrow{\div(-7)} y = 2$$

In fact, we can now use the same trick to find x by eliminating y:

$$\begin{cases} 2x - 5y &= -4 \\ 3x - 4y &= 1 \end{cases} \begin{array}{c} \xrightarrow{\times(-4)} \\ \xrightarrow{\times 5} \end{array} \begin{cases} -8x + 20y = 16 \\ 15x - 20y = 5 \end{cases} \xrightarrow{\text{add equations}} 7x = 21 \xrightarrow{\div 7} x = 3$$

Thus, the solution is $(3, 2)$.

What numbers shall we use to multiply each equation? Something that always works if we want to eliminate x is to multiply the first equation by the coefficient of x in the second equation, and the second by minus the coefficient of x in the first. Same thing if we want to eliminate y: multiply each equation by the coefficient of the other, and then change the sign of one of them. There are other choices, as in the example below, but this one always works.

Example: Solve $\begin{cases} 3x - 2y = 2 \\ -2x + 4y = 2 \end{cases}$

Let us find x by eliminating y. Multiply the first equation by 4 and the second by 2, add the equations, and solve the equation you get:

$$\begin{cases} 3x - 2y & = 2 \\ -2x + 4y & = 2 \end{cases} \begin{array}{c} \xrightarrow{\times 4} \\ \xrightarrow{\times 2} \end{array} \begin{cases} 12x - 8y = 8 \\ -4x + 8y = 4 \end{cases} \xrightarrow{\text{add equations}} 8x = 12 \xrightarrow{\div 8} x = \frac{12}{8} = \frac{3}{2}$$

Now let us eliminate x:

$$\begin{cases} 3x - 2y & = 2 \\ -2x + 4y & = 2 \end{cases} \begin{array}{c} \xrightarrow{\times 2} \\ \xrightarrow{\times 3} \end{array} \begin{cases} 6x - 4y = 4 \\ -6x + 12y = 6 \end{cases} \xrightarrow{\text{add equations}} 8y = 10 \xrightarrow{\div 8} y = \frac{10}{8} = \frac{5}{4}$$

Therefore the solution of the system is $(\frac{3}{2}, \frac{5}{4})$.

Practice exercises: Solve the following systems

5. $\begin{cases} 3x - 2y = 2 \\ -2x + y = 3 \end{cases}$ **6.** $\begin{cases} -2x + 2y = -1 \\ x + y = 2 \end{cases}$ **7.** $\begin{cases} 3x - y = 3 \\ 2x + 5y = 2 \end{cases}$ **8.** $\begin{cases} x + y = -5 \\ 2x + y = 1 \end{cases}$

9. $\begin{cases} 6x - 3y = 1 \\ 2x + 3y = 3 \end{cases}$ **10.** $\begin{cases} -3x + 2y = 3 \\ 4x + 3y = 2 \end{cases}$ **11.** $\begin{cases} 2x - 3y = 4 \\ -4x + 6y = 2 \end{cases}$ **12.** $\begin{cases} 3x + y = 3 \\ -6x - 2y = -6 \end{cases}$

Types of solutions

Normally a system of equations has a unique solution. However, there are two other possibilities, as in the last two exercises:

- The system has **only one solution**. This is what we have seen in all the exercises except the last two.
- The system has **no solutions**. This happens when the two equations are giving contradictory information, such as $\begin{cases} x + y = 3 \\ x + y = 4 \end{cases}$. If $x + y = 3$, then it is impossible that $x + y = 4$, so there cannot be any solutions. These systems are called *inconsistent*.

  You will see that a system has no solutions when, in the process of solving it, you get an equation with no variables that is not true (like $3 = 4$ in the example above).
- The system has a **whole line of solutions**. This when the two equations are giving exactly the same information, such as $\begin{cases} x + y = 3 \\ x + y = 3 \end{cases}$. The first equation says that $x+y = 3$, but the second equation says exactly the same, so in reality we do not have a system of equations, we have a single equation. The set of solutions of this equation is the set of solutions of either of the two equations (both equations have the same set of solutions), in this case the line $x + y = 3$, which can be represented with a graph, as in the previous section.

  You will see that a system has a line of solutions when, in the process of solving it, you get an equation with no variables that is true (like $3 = 3$ in the example above).

  Practice exercises: Solve the following systems

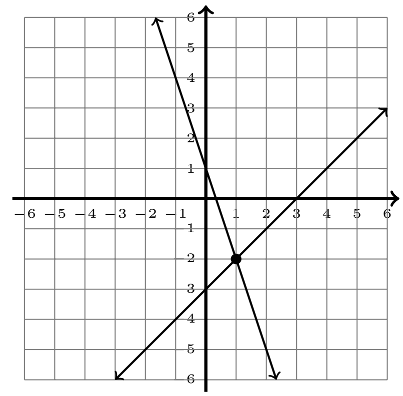
  13. $\begin{cases} x - 2y = 2 \\ 3x - 6y = 3 \end{cases}$ **14.** $\begin{cases} 2x - y = 1 \\ -6x + 3y = -3 \end{cases}$ **15.** $\begin{cases} 3x - 2y = 3 \\ 2x + y = 1 \end{cases}$

Systems and graphing

Recall from section 3.5 that, to represent the set of solutions of a single linear equation in two variables, we used a graph, and that the solutions formed a straight line.

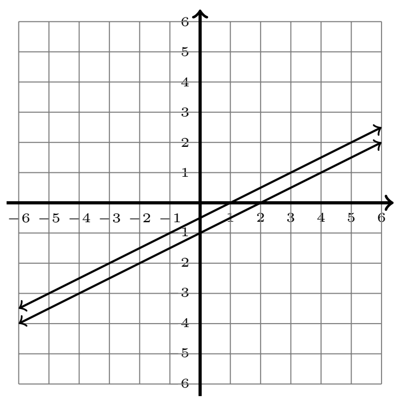
When we have a system we have two equations, so we have two lines instead of one, and we want to find a solution of *both* equations at the same time, so **the solution of a system is the point where the lines representing the solutions of the equations intersect**.

For example, we saw above that the point $(1, -2)$ is a solution of the system $\begin{cases} x - y = 3 \\ 3x + y = 1 \end{cases}$.
If we graph these equations we get the following picture, where we see that the solution, $(1, -2)$, is exactly the point where the lines intersect.



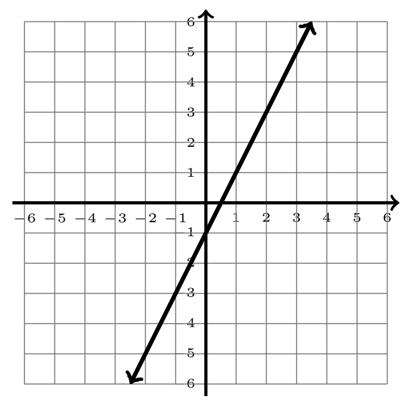
This gives a graphical interpretation of what happens when a system has no solutions and when a system has a whole line of solutions:

For example, in exercise 13 we saw that the system $\begin{cases} x - 2y = 2 \\ 3x - 6y = 3 \end{cases}$ has no solutions.
If we graph the lines we find out why:



The lines are parallel! Thus, they do not intersect.

For example, in exercise 14 we saw that the system $\begin{cases} 2x - y = 1 \\ -6x + 3y = -3 \end{cases}$ has a whole line of solutions.
If we graph the lines we find out why:



The two lines coincide, so they have the same graph. Thus, the whole line is the set of solutions.

Equations in more general form

In all the examples we have seen, terms with variables appear on the left, and number terms appear on the right. What do we do if this is not the case? Simply use the rules to manipulate equations to write them as before and then use the methods above. For example: Solve $\begin{cases} 2x = 2 + 3y \\ 3x + 3 = -y \end{cases}$

For the first equation, subtract $3y$ from both sides to get $2x - 3y = 2$. For the second equation, subtract 3 from both sides and then add y to both sides to get $3x + y = -3$. We get the system $\begin{cases} 2x - 3y = 2 \\ 3x + y = -3 \end{cases}$ which we can solve as before.

<u>Practice exercises</u>: Solve the following systems

16. $\begin{cases} -2y = 2 + x \\ -2x = 3 - 2y \end{cases}$ 1 **17.** $\begin{cases} 2y = 3(-1 + x) \\ x - 2 = y + 3 \end{cases}$ **18.** $\begin{cases} x = y \\ y = x \end{cases}$ **19.** $\begin{cases} 3x = 3 + y \\ 2x + 5y = -2 \end{cases}$