3.1. Graph linear equations in two variables. Professor Luis Fernández

Linear equations in two variables

Recall: A *solution* of a linear equation in two variables, such as $3x + 2y = 6$, is a pair of numbers (a, b) such that, when we substitute $x = a$, $y = b$ we obtain a true statement. For example, $(2, 0)$ is a solution of $3x + 2y = 6$ because if $x = 2$ and $y = 0$, then $3 \cdot 2 + 2 \cdot 0 = 6 + 0 = 6$.

Linear equations have infinitely many solutions. For example, you can check that $(0, 3)$, $(4, -3)$, $(6, -6)$ are all solutions of the equation $3x + 2y = 6$ above.

To find solutions of linear equations in two variables, there are different methods:

- Trial and error. Often it is easy to spot a solution by looking at the equation. This is how the solutions above were found.
- Put one variable equal to 0 and solve for the other variable. Sometimes you will get a fraction as the coordinate of the other variable, but that is fine. For example, if we write $x = 0$ in the equation $2x + 3y = 5$, we get $2 \cdot 0 + 3y = 5 \quad \to \quad 3y = 5 \quad \to \quad y = 5/3$. Therefore $(0, 5/3)$ is a solution.
- If one variable is solved for, as for example $y = 2x - 3$, simply substitute values of x and find the corresponding value of y. For example, if $x = 0$, $y = -3$; if $x = 1$, $y = -1$. Thus, $(0, -3)$ and $(1, -1)$ are solutions.
- Finally one can solve for one of the variables and then use the last method. For example, in $2x - 5y = 6$, we can solve for y to get $-5y = -2x + 6 \quad \to \quad y = 2/5x - 6/5$. Then substitute values of x to get the corresponding value of y.

Practice exercises: Find 3 solutions of each of the following equations.

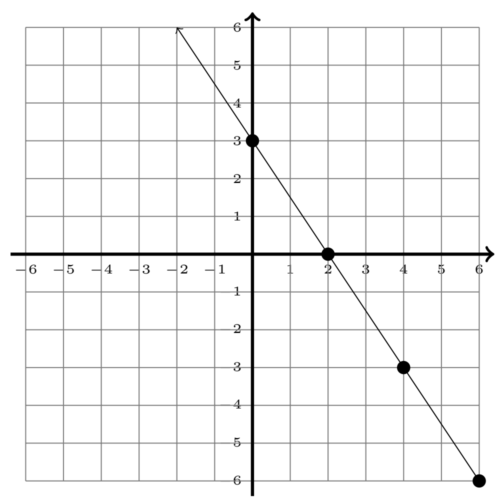
1. $y = 3x - 2$ **2.** $2x + 5y = 10$ **3.** $x + y = 0$

4. $-2x + 3y = 5$ **5.** $x = 3y - 4$ **6.** $3x + 4y = 5$

Graphing the set of solutions of a linear equation in two variables

Since the set of solutions of a linear equation is infinite, we cannot just list all the solutions. Instead, we do a graph that tells us what the solutions are. Recall that a pair of numbers, such as $(3, 4)$, corresponds to a point in the Cartesian coordinate system:

If one plots all the pairs that are solutions to a linear equation, one obtains a line. For example, if we plot the solutions of $3x + 2y = 6$ that we found above (that is, $(2, 0)$, $(0, 3)$, $(4, -3)$, $(6, -6)$) we get the dots in the graph to the right. If we found *every* possible solution, it makes sense to think that they will lie in a straight line, as shown.

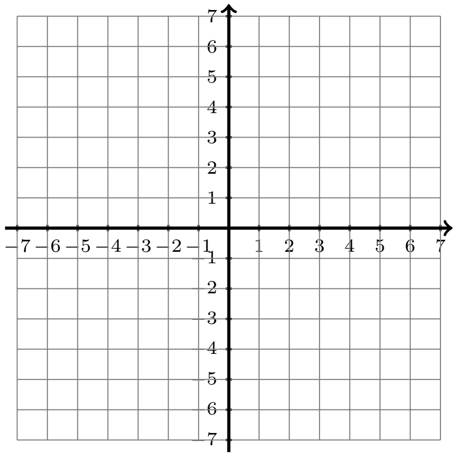


Thus, to graph the set of solutions of a linear equation in two variables, the first method we see is as follows:

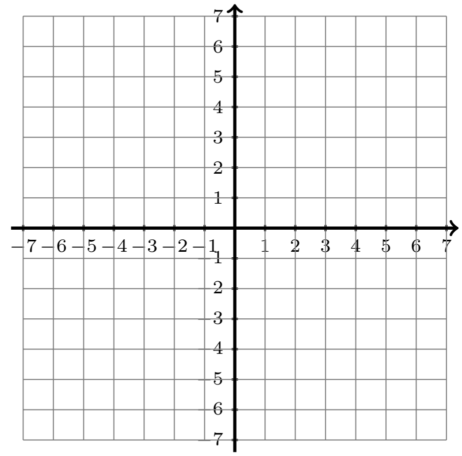
- Find three pairs of numbers that are solutions of the given equation. Two points would be enough, but it is better to have 3 points in case one makes a mistake.
- Plot the points and graph the (infinite) line passing through these points. If the 3 points do not lie on a line you have made a mistake. Check your solutions.

Practice exercises: Graph the set of solutions of each of the following equations.

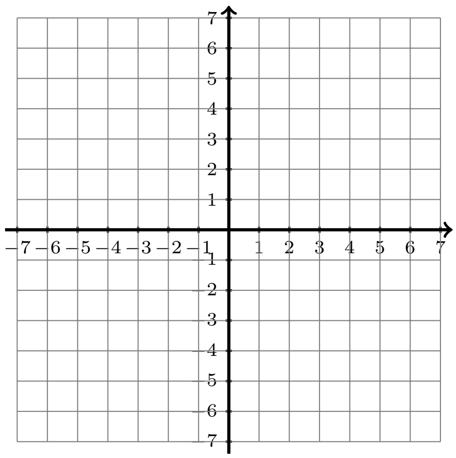
7. $y = 3x - 2$



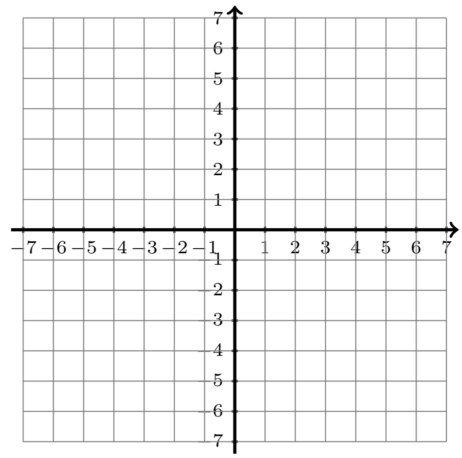
8. $2x + 5y = 10$



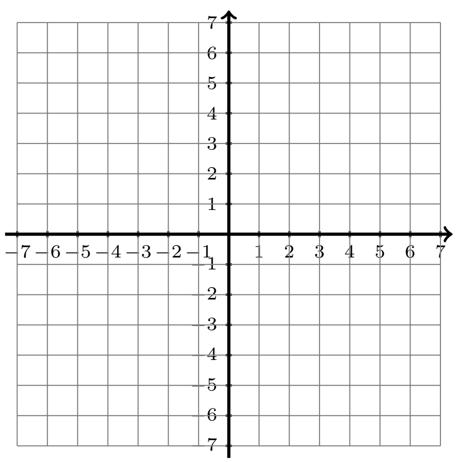
9. $x + y = 0$



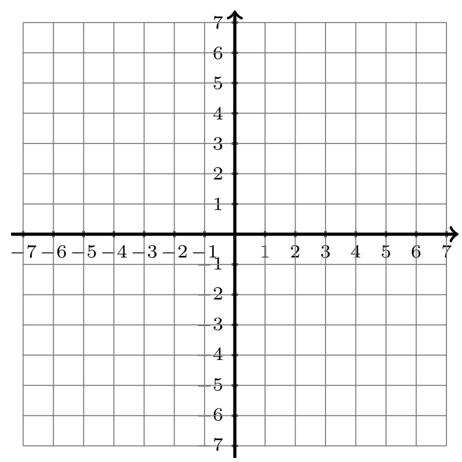
10. $-2x + 3y = 5$



11. $x = 3y - 4$



12. $3x + 4y = 5$



Vertical and horizontal lines

It is possible to have an equation in two variables in which one of the variables does not appear at all. For example, $x = 5$, or $y = -3$, or $x - 3 = 1$ are of this kind.

Notice that when we make a table of values to find solutions, there is no information about one of the variables. For example, to find solutions of $x = 3$, which value of y shall we take?

Well, since the equation $x = 3$, says nothing about y, this means that y can be anything we like. So for example, $(3, 4)$, $(3, 1034)$, and $(3, -23)$ are all solutions. In fact, as long as the first coordinate is 3, it will be a solution.

Likewise, for the equation $y = 4$, $(3, 4)$, $(-23, 4)$, $(0, 4)$ are all solutions, as would any pair of numbers as long as the second number is 4.

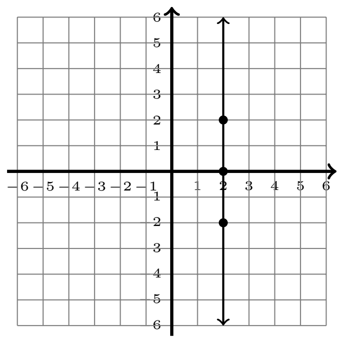
Linear equations in two variables that miss a variable can be graphed the same way as before: do a table of values, plot 3 points, and draw the line. One notices, however, that

- Equations of the form $x = a$ are *vertical lines* passing through $(a, 0)$.
- Equations of the form $y = b$ are *horizontal lines* passing through $(0, b)$.

Example: Graph the equation $x = 2$.

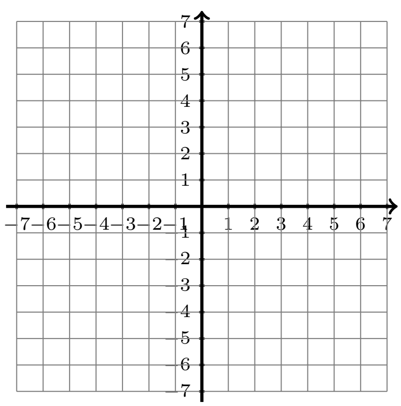
Let us find 3 solutions. We can do a table of values and then plot the points as follows:

x	y
2	0
2	2
2	-2

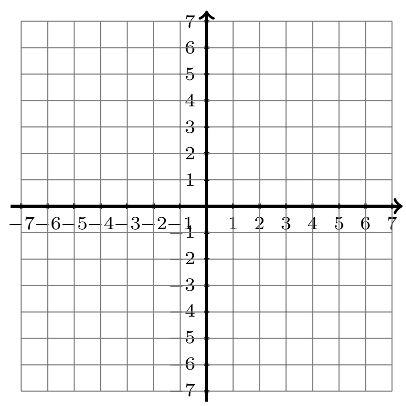


Practice exercises: Graph the set of solutions of each of the following equations.

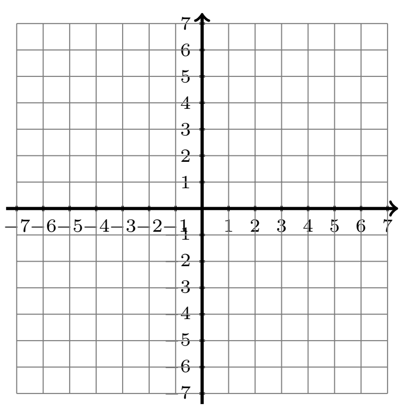
13. $y = -2$



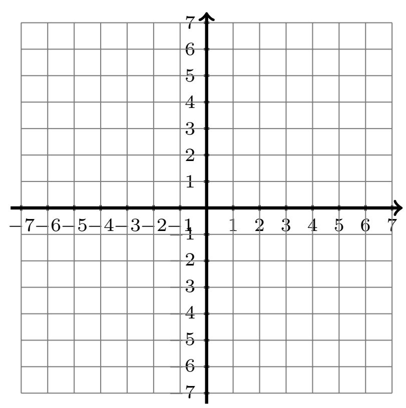
14. $x = 5$



15. $x = 0$



16. $y = 0$



Intercepts

The points where a line intercepts the axes are called *intercepts*:

- The x-intercept is the point where a line intercepts the x-axis.
- The y-intercept is the point where a line intercepts the y-axis.

Note that

- The y-coordinate of the points that lie in the x-axis is 0. Thus, the x-intercept has the form $(\text{number}, 0)$.
- The x-coordinate of the points that lie in the y-axis is 0. Thus, the y-intercept has the form $(0, \text{number})$.

Hence,

- To find the x-intercept of a line, put $y = 0$ in the equation and solve for x.
- To find the y-intercept of a line, put $x = 0$ in the equation and solve for y.

Example: Find the x and y intercepts of the line with equation $3x + 4y = 6$.

x-intercept $\rightarrow$ Put $y = 0$ and find x: $3x + 4 \cdot 0 = 6 \;\rightarrow\; 3x = 6 \;\rightarrow\; x = 2$.

y-intercept $\rightarrow$ Put $x = 0$ and find y: $3 \cdot 0 + 4y = 6 \;\rightarrow\; 4y = 6 \;\rightarrow\; y = 6/4 = 3/2$.

Thus the x-intercept is $(2, 0)$ and the y-intercept is $(0, 3/2)$.

Practice exercises: Find the intercepts of the lines given by the following equations.

17. $2x + 3y = 6$	**18.** $-4x + y = 4$	**19.** $x + 4y = 0$
20. $y = 6$	**21.** $x = 4$	**22.** $y = 2x + 5$
23. $y = 6x - 2$	**24.** $y = 4x + 10$	**25.** $y = -x + 5/3$

Slope

Given a linear equation in two variables, notice that there is a pattern when we find the x and corresponding y values. For example, for the equation $3x - 2y = -1$, we could do the following table of values:

x	1	3	5
y	2	5	8

One can see that when x goes up 2, y goes up 3. Another pair of values would be $(7, 11)$. And then another would be $(9, 14)$. We can also go down, so another would be $(-1, -1)$.

The ratio between the change in y and the corresponding change in x is called the *slope* of the line. Thus, the slope of $3x - 2y = -1$ is $3/2$ (because as y goes up 3, x goes up 2).

Given two points on the line, say (x_1, y_1) and (x_2, y_2), the change in the x coordinate from one point to the other is $(x_2 - x_1)$, and the change in the y coordinate will be $(y_2 - y_1)$. Therefore the slope (usually denoted with the letter m) of a line passing through two points (x_1, y_1) and (x_2, y_2) is

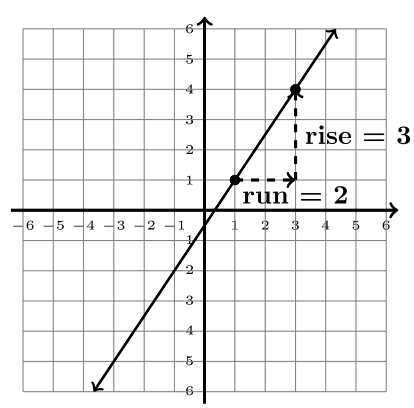
$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope gives a measure of how *steep* a line is as you move along it, going left to right. To find the slope of a line from the picture, find any two points on the line (in this case, $(1, 1)$ and $(3, 4)$). Then find how much the y coordinate of the two points changes (usually called the *rise*; it is 3 in this case) and divide it by the change in the x coordinate (usually called the *run*; it is 2 in this case). Thus,

$$m = \frac{\text{rise}}{\text{run}},$$

and for this example,

$$m = \frac{3}{2}.$$

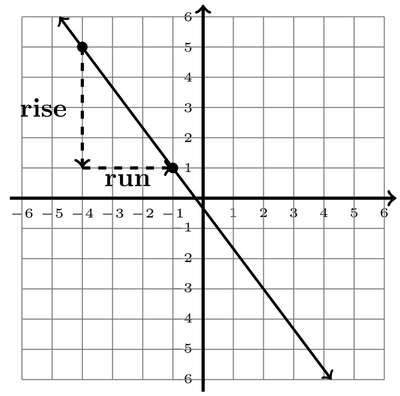


NOTE: A line can also have **negative** slope (which means that we go downhill as we move along the line, going left to right. For example:

Rise $= -4$ (because it goes down).

Run $= 3$ (because it goes to the right).

$$m = \frac{-4}{3} = -\frac{4}{3}.$$



NOTE: The slope of a vertical line is undefined (because the change in x is 0, and division by 0 is undefined). The slope of a horizontal line is 0 (because the change in y is 0, and 0 divided by anything is 0).

Practice exercises:

26. Find the slope of the lines in the picture (labeled by letters).

a) $m =$

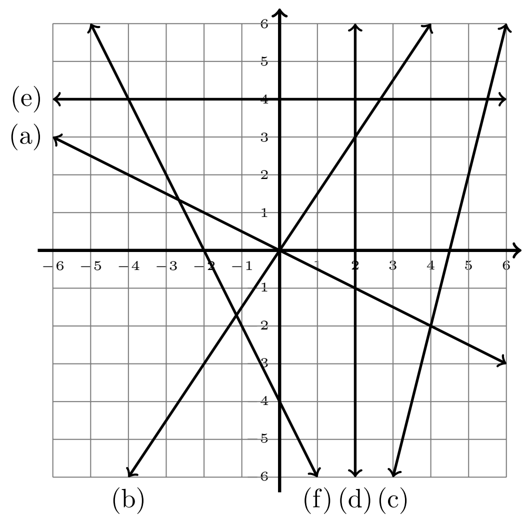
b) $m =$

c) $m =$

d) $m =$

e) $m =$

f) $m =$



Finding the slope from the equation: What if we want to find the slope from the equation? We could find two points that satisfy the equation and then find it as before. But there is another way. To find the slope of a line given its equation,

- Solve the equation for y to write it in the form $y = mx + b$.
- Then m is the slope and b is the y-intercept.

Example: Find the slope and y intercept of the line with equation $3x + 2y = 5$.

Solve for y: $3x + 2y = 5 \quad \rightarrow \quad 2y = -3x + 5 \quad \rightarrow \quad y = -\frac{3}{2}x + \frac{5}{2}$.

Therefore the slope is $-\frac{3}{2}$ and the y intercept is $\frac{5}{2}$.

Practice exercises Find the slope and the y intercept of the lines that have the following equations.

27. $y = 3x - 2$ **28.** $3x + y = 4$ **29.** $3x - 5y = 4$

30. $y = 2$ **31.** $x = 4$ **32.** $4x - y = 4$

33. $x - 3y = -3$ **34.** $x + y = 4$ **35.** $3x - 7y = -1$

3.5. Relations and functions. Professor Luis Fernández

A function is essentially a machine that receives an input and produces an output. Functions are used thoroughout in science, computers, technology, business, and essentially any human activity. For example, anything you do in your phone involves lots of functions for the phone to do what you want given your input.

The important thing about a function is that each input produces only one output. Otherwise the result would be unpredictable!

A common way to represent a function is using *functional notation.* First, the function needs to have a name. In mathematics, just to make it short, one often uses letters like f, g, or h to name the function, but in programming, for example, it is better to use names as `sendMessage`, because programs often contain thousands of functions and it is better to have a descriptive name that one remembers. Second, we use a *dummy variable*, which could be anything at all. In math one often uses x, but t, s or y are also common. The name of the variable is not important. We write the dummy variable in parenthesis, right after the name of the function. When one defines a function, on the right hand side one has the formula that one has to apply to the dummy variable to get the value of the function. For example, the function that takes a number and multiplies it by 7, would be written as:

Parenthesis (they do not mean anything, but have to be there)

$$f(x) = 7x$$

Name of the function

Name of the dummy variable

Formula in terms of the variable

By *evaluating* a function we mean substituting the value of the variable into the formula. For example if, as above, $f(x) = 7x$, then $f(3) = 21$, $f(-1) = -7$, etc.

Example: If $g(t) = t + 5$, find $g(3)$, $g(-4)$, $g(p)$, $g(x^2 + 1)$, $g(3s + 2)$.

- $g(3) = 3 + 5 = 8$ (write 3 wherever the dummy variable, t in this case, appears in the definition).
- $g(-4) = (-4)+5 = 1$ (write (-4) wherever the dummy variable, t in this case, appears in the definition).
- $g(p) = p + 5$ (write p wherever the dummy variable, t in this case, appears in the definition).
- $g(x^2 + 1) = (x^2 + 1) + 5 = x^2 + 6$ (write $(x^2 + 1)$ wherever the dummy variable, t in this case, appears in the definition).
- $g(3s + 2) = (3s + 2) + 5 = 3s + 7$ (write $3s + 2$ wherever the dummy variable, t in this case, appears in the definition).

Practice exercises: Evaluate the following functions.

36. $h(3)$, $h(0)$, and $h(-2)$, if $h(y) = 2y - 2$.

37. $R(5)$, $R(-3)$, and $R(t)$, if $R(s) = s^2 + s + 1$

38. $f(5)$, $f(x + 3)$, and $f(x^2 - 1)$ if $f(x) = \dfrac{x+1}{x-3}$

39. $h(-1)$, $h(hop)$, and $h(blah)$ if $h(x) = 3x + 7$

40. $L(3)$, $L(x^2 + 3)$, and $L(x^2)$ if $L(u) = \dfrac{u-3}{u+2}$

41. $T(run)$, $T(x - 1)$, and $T(-9)$ if $T(x) = 2x^2 - 3x + 2$.

NOTE for the future: Given a function, for example $f(x) = x^2 + 1$ we can do a table of values as follows:

x	$f(x)$
-2	5
-1	2
0	1
1	2
2	5

When we plot these points (and many more), the result is a *parabola*, as in the picture.