2.1. Use a General Strategy to Solve Linear Equations. Professor Luis Fernández

Combining like terms

Recall: In a mathematical expression with variables, the parts of the expression separated by a "+" or a "−" are called *terms*.

Two terms are *like terms* if they contain the same variables with the same exponent (or they have no variables at all). For example, in the expression

$$2x + 4 - 3x + 6 - x$$

$2x, -3x$, and $-x$ are like terms, and so are 4 and 6.

Like terms can be combined to make the expression simpler. To do this, just add the numbers multiplying the variable (with their appropriate sign) and leave the variable the same. For example,

$$2x + 4 - 3x + 6 - x = (2x - 3x - x) + (4 + 6) = -2x + 10.$$

Note that **unlike terms cannot be combined**. For example, unless I know the value of x, I cannot combine x and 4 in the expression $x + 4$.

Practice exercises: Combine like terms.

1. $3x + 5x + 7 - 2 =$

2. $-x - 6 + 12x + 5 - 7x =$

3. $10x - 6x + 4 - 3x =$

4. $3x - 5 - 4 + 8x - 3x + 7 =$

Equations and solutions

Recall: An **equation** is a mathematical statement with an "=" and one of more variables (that is, unknown numbers) represented by letters.

Examples: $2x + 3 = 7$, $3x - 5 = 6x + 7$, $2x + y = 12$, $x^2 - 6x + 8 = 0$ are all equations.

A **solution** of an equation is a value of the variables that make the equation an *equality* (that is, a *true* statement with an "=" and no variables).

For example: Is $x = 2$ a solution of $2x + 3 = 7$? Let us substitute $x = 2$ in the equation. To make it easier, let us substitute on each side of the equation separately:

- Substitute $x = 2$ on the left hand side (LHS) of the equation ($2x+3$) and simplify: LHS$= 2\cdot(2)+3 = 4+3 = 7$.
- Substitute $x = 2$ on the right hand side (RHS) of the equation (7) and simplify: we just get RHS$= 7$ (there is no x).

Since the LHS is equal to the RHS (both sides equal 7), $x = 2$ *is* a solution of $2x + 3 = 7$.

For example: Is $x = 3$ a solution of $3x - 5 = 6x + 7$?

- Substitute $x = 2$ on LHS: $3 \cdot (3) - 5 = 9 - 5 = 4$.
- Substitute $x = 2$ on HS: $6 \cdot (3) + 7 = 18 + 7 = 25$.

Since the LHS and the RHS are not equal, $x = 3$ *is not* a solution of $3x - 5 = 6x + 7$.

For example: Is $x = (-4)$ a solution of $3x - 5 = 6x + 7$?

- Substitute $x = (-4)$ on LHS: $3 \cdot (-4) - 5 = (-12) - 5 = (-17)$.
- Substitute $x = (-4)$ on HS: $6 \cdot (-4) + 7 = (-24) + 7 = (-17)$.

Since the LHS and the RHS are equal (they are both (-17)), $x = (-4)$ *is* a solution of $3x - 5 = 6x + 7$.

Exercises: Determine whether the given value of the variable is a solution of the given equation.

5. $x = 2$, of $2x + 6 = 10$.

6. $x = 4$, of $2x - 2 = -5$.

7. $x = (-1)$, of $2(x + 4) = 3(x + 3)$.

8. $x = 5$, of $4(x - 2) = 4x - 8$.

9. $x = (-2)$, of $2x + 6 = 3x + 8$.

10. $x = 4$, of $3x + 6 = 5x - 2$.

Solving simple equations

Recall: An equation is called *linear* if it has no exponents. To solve a linear equation, we manipulate the equation in order to simplify it and obtain the value of the variable. The rules to manipulate equations are as follows:

1. You can add **the same** number or expression to **both sides** of the equation.

2. You can subtract **the same** number or expression from **both sides** of the equation.

3. You can multiply **both sides** of the equation by **the same** number or expression.

4. You can divide **both sides** of the equation by **the same** number or expression.

You can use *any* of these rules as many times as you want. Just remember: it is easier to use rules 1 and/or 2 first, and then finish with 3 and/or 4.

Example: Solve the equation $3x + 4 = 10$.

- Subtract 4 from both sides of the equation and combine like terms: $3x + 4 - 4 = 10 - 4 \Rightarrow 3x = 6$.
- Divide both sides by 3: $3x \div 3 = 6 \div 3 \Rightarrow x = 2$ (because $3x \div 3 = x$).

Therefore the solution is $x = 2$. It easily checks: $3 \cdot (2) + 4 = 6 + 4 = 10$.

The goal is to leave x alone in one of the sides of the equation to end up with an expression of the form $x =$ number or number $= x$. The general idea to do this is:

a. If necessary, combine like terms in each side of the equation separately.

b. Subtract the constant term of the LHS from **both** sides of the equation. This way the LHS will have no constant term (you *moved it* to the other side).

c. Subtract the linear term of the RHS from **both** sides. This way the RHS will have no linear term (you *moved it* to the other side).

d. Divide **both** sides by the coefficient of the linear term in the LHS.

Example: Solve the equation $3x + 6 = x + 11$.

- Subtract 6 (the constant term of the LHS) from both sides of the equation and combine like terms:
  $3x + 6 - 6 = x + 11 - 6 \Rightarrow 3x = x + 5$.
- Subtract x (the linear term of the RHS) from both sides of the equation and combine like terms:
  $3x - x = x + 11 - x \Rightarrow 2x = 11$.
- Divide both sides by 2 (the coefficient of the linear term on the LHS): $2x/2 = 11/2 \Rightarrow x = 11/2$

Therefore the solution is $x = \frac{11}{2}$.

Example: Solve the equation $\dfrac{3x}{2} + 5 = 11$.

- Since the linear term on the LHS has a denominator (2), multiply both sides by 2 and simplify:
  $2 \cdot \dfrac{3x}{2} = 2 \cdot 6 \Rightarrow \dfrac{6x}{2} = 12 \Rightarrow 3x = 12$
- Divide both sides by 3 (the coefficient of the linear term on the LHS): $3x/3 = 12/3 \Rightarrow x = 4$.

Therefore the solution is $x = 4$.

Exercises: Solve the following equations.

11. $2x + 7 = 15$

12. $5x - 4 = 18$

13. $7x - 5 = 12$

14. $6 - 2x = 14$

15. $-8 - 7x = -1$

16. $-5x + 7 = 12$

17. $6x - 5 = 2x - 13$

18. $4x + 2 = 2 + x$

19. $4 = 2 + x$

20. $\dfrac{x}{5} + 6 = 9$

21. $\dfrac{5x}{2} = 15$

22. $\dfrac{-4x}{3} = -16$

23. $\dfrac{-2x}{3} = 8$

24. $\dfrac{x}{2} - 4 = 7$

25. $\dfrac{5x}{3} + 6 = -5$

Solving more complicated equations

Recall: When equations have expressions involving parenthesis, one needs to use the *distributive property* first. For example, to solve

$$2(x-4)+5=3-3(x-1)+6,$$

first distribute the parenthesis: $2(x-4)=2x-8$ and $-3(x-1)=-3x+3$, to get

$$2x-8+5=3-3x+3+6.$$

Then, as before, combine like terms

$$2x-3=-3x+12,$$

Add 3 to both sides, then add $3x$ to both sides:

$$2x+3x=+15, \text{ so that} 5x=15,$$

and finally divide both sides by 5 to get $x=3$.

Practice exercises:

26. $5(x-4)=18$ **27.** $-(x-5)+4=12$ **28.** $2(x+7)-4=15-2(x-3)$

29. $6-2(x+3)=14-3(x+1)$ **30.** $8-(x+3)=9-(x+4)$ **31.** $-5(x+3)=4-5x$

Types of equations

Some equations have only one solution. For example, the ones in the exercises above have only one solution. These are called *conditional equations*.

But sometimes strange things happen. For example, look at the equation $x+1=x+1$. Well, $x+1$ is always equal to $x+1$, no matter what x is. In other words, every value of the variable is a solution of the equations. These types of equations are called *identities*.

And sometimes it turns out that the equation has no solutions. For example, $x=x+3$ cannot be possible (no number is equal to itself plus 3!). These equations are called *contradictions*.

To find out what kind of equation it is, just solve the equation. At the end you will get one of these possibilities:

- You get something like "$x=$ number" (or "number $=x$"). Then there is only one solution and the equation is *conditional*.
- You get something like "number = same number" (with no variable). This statement is always true for any value of the variable (for example, $3=3$ is always true). Then the equation is an *identity*.
- You get something like "number = different number" (with no variable). This statement is never true for any value of the variable (for example, $3=5$ is never true). Then the equation is a *contradiction*.

• You get something like "number = different number" (with no variable). This statement is never true for any value of the variable (for example, $3=5$ is never true). Then the equation is a *contradiction*.

Example: Classify the equation $2(x-1)+3=2x-5$.

Let us solve the equation: $2(x-1)+3=2x-5 \quad\rightarrow\quad 2x-2+3=2x-5 \quad\rightarrow\quad 1=-5$, which is impossible. Therefore it is a *contradiction*.

Practice exercises: Classify the following equations as conditional, contradictions or identities.

32. $5(x-1)+3=5x-2$ **33.** $-(x-2)+4=7-x$ **34.** $2(x+1)-1=3-x$

35. $6x-2(x+3)=-(4x+6)$ **36.** $2-(x+1)=5-2(x-1)$ **37.** $-2(x-2)+2(x-2)=1$

Solving equations that contain denominators

Recall: An equation that contains denominators can be solved using the same techniques as above, but then we would have to do operations with fractions. There is a more efficient method, as follows:

- Write all the fractions in the equation with a common denominator. *This includes those terms in the fraction that have no denominator; write it as a fraction with a 1 as denominator first.*
- Once all the terms have the same denominator, *remove all the denominators and write each numerator in parenthesis.*
- Solve the equation as you did before.

Example: Solve the equation $\dfrac{x-1}{3}+4=\dfrac{x+2}{5}$.

Let us write it first so that *every term has a denominator* (write 1 under the 4): $\dfrac{x-1}{3}+\dfrac{4}{1}=\dfrac{x+2}{5}$.
The denominators are $3, 5$ and 1, so the common denominator is 15. Multiply numerator and denominator of each fraction by the appropriate number so that they all have 15 as denominator:

$$\frac{5(x-1)}{15}+\frac{15\cdot 4}{15}=\frac{3(x+2)}{15}, \quad \text{which gives} \quad \frac{5x-5}{15}+\frac{60}{15}=\frac{3x+6}{15}.$$

Now remove the denominators and write each numerator in parenthesis to get

$$(5x-5)+(60)=(3x+6).$$

This is easy to solve:

$$(5x-5)+(60)=(3x+6) \quad \rightarrow \quad 5x+55=3x+6 \quad \rightarrow \quad 2x=-49 \quad \rightarrow \quad \boxed{x=-\frac{49}{2}}.$$

Example: Solve the equation $x+2-\dfrac{x-1}{4}=\dfrac{2x}{6}$.

Let us write it first so that *every term has a denominator* (write 1 under the 4): $\dfrac{x+2}{1}-\dfrac{x-1}{4}=\dfrac{2x}{6}$
The denominators are $1, 4$ and 6, so a common denominator is 12. Multiply numerator and denominator of each fraction by the appropriate number so that they all have 15 as denominator:

$$\frac{12(x+2)}{12}-\frac{3(x-1)}{12}=\frac{2\cdot 2x}{12}, \quad \text{which gives} \quad \frac{12x+24}{12}-\frac{3x-3}{12}=\frac{4x}{12}.$$

Now remove the denominators and write each numerator in parenthesis to get

$$(12x+24)-(3x-3)=(4x).$$

This is easy to solve:

$$(12x+24)-(3x-3)=(4x) \quad \rightarrow \quad 12x+24-3x+3=4x \quad \rightarrow \quad 5x=-27 \quad \rightarrow \quad \boxed{x=-\frac{27}{5}}.$$

Practice exercises:

38. $\dfrac{x-3}{4}+\dfrac{x+1}{5}=\dfrac{x+1}{10}$

39. $3-\dfrac{x-1}{6}=4+\dfrac{x+1}{3}$

40. $\dfrac{-2x+1}{15}-\dfrac{x}{5}=2$

41. $\dfrac{x-1}{15}=x-\dfrac{x-1}{10}$

42. $\dfrac{x}{4}+\dfrac{1}{3}=\dfrac{x+1}{8}$

43. $1-\dfrac{x-1}{7}=4-\dfrac{x+1}{7}$