

CSI 35: Discrete Mathematics II. Midterm 1, Version B

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NAME: _____

INSTRUCTIONS:

- This exam contains 9 questions. 6 pages for a total of 104 points (points over 100 count as extra credit).
- You have 110 minutes to complete the exam.
- **You must show all your work** in order to get credit.
- You can use a non-graphing scientific calculator. No other electronic devices, notes or books are permitted.

1. [10 points] We recursively define a function f from the positive integers to the integers as follows: $f(1) = -1$ and $f(n) = 2 \cdot f(n-1) + 5$.

Evaluate the following.

(a) $f(1) = -1$
(given)

(b) $f(2)$

$$\begin{aligned} f(2) &= 2f(1) + 5 \\ &= 2(-1) + 5 \\ &= 3 \end{aligned}$$

(c) $f(3)$

$$\begin{aligned} f(3) &= 2f(2) + 5 \\ &= 2(3) + 5 \\ &= 11 \end{aligned}$$

2. [12 points] Let $F(n) = -4n$, defined on integers $n \geq 1$. Find a *recursive* definition for F .

Function $F(n)$:

$$F(1) = -4$$

$$F(n) = F(n-1) - 4 \text{ if } n > 1.$$

3. [12 points] Consider the following program:

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if  $x < 0$ :  
     $x = x + 3$   
else:  
     $x = 2x + 1$ 
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Suppose you are given that x is initialized to some positive integer. Show that after running the program, the value of x is odd.

If $x > 0$, then program will skip the "if $x < 0$ " and jump to the "else",

giving $x = 2x + 1$. No matter what the initial x is, $2x + 1$ is an odd number, so the final value of x will be odd.

$\begin{matrix} \uparrow & \uparrow \\ \text{new} & \text{old} \\ \text{value} & \text{value} \\ \text{of } x & \text{of } x \end{matrix}$

4. [12 points] Give a recursive definition of the set of all positive integers that are multiples of 5.

Set S :

$5 \in S$

if $x \in S$ and $y \in S$ then $x + y \in S$.

5. [10 points] Consider the proposition: 3 divides $n^3 - n$ for $n \geq 2$. Suppose that we want to prove this proposition by induction.

(a) State the propositional function $P(n)$ that you want to prove.

$P(n)$ is: 3 divides $n^3 - n$.

(b) State and prove the BASIC STEP.

BASIC STEP: $P(2)$ is "3 divides $2^3 - 2 = 6$ ".
which is true: $3 \cdot 2 = 6$.

(c) State the INDUCTIVE STEP. Do not prove it.

If 3 divides $n^3 - n$, then 3 divides $(n+1)^3 - (n+1)$.

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6. [12 points] Write a recursive algorithm called " $a(n)$ ", for n a nonnegative integer, that computes the n^{th} term of the sequence a_n defined by $a_0 = 1$, $a_1 = 3$, and $a_n = 3a_{n-1} + a_{n-2}$. You should write it as follows:

procedure $a(n$: n nonnegative integer)

if $n = 0$, return 1

else if $n = 1$ return 3

else return $3a(n-1) + a(n-2)$

7. [12 points] Let r be a real number. Prove by induction that $r^0 + r^1 + r^2 + \dots + r^n = \frac{r^{n+1} - 1}{r - 1}$.
Make sure to clearly state $P(n)$ and clearly explain the base and inductive steps.

$$P(n) \text{ is: } r^0 + \dots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

BASIC STEP

$$P(0) \text{ is: } \begin{array}{ccc} r^0 & = & \frac{r^{0+1} - 1}{r - 1} \\ \parallel & & \parallel \\ 1 & & \frac{r - 1}{r - 1} = 1 \end{array} \quad \checkmark P(0) \text{ is true.}$$

INDUCTIVE STEP

Assume that, for some $n \geq 1$, $P(n)$ is true, that is, $r^0 + \dots + r^n = \frac{r^{n+1} - 1}{r - 1}$ (ind. hyp.)

Add r^{n+1} to both sides to get

$$r^0 + \dots + r^n + r^{n+1} = \frac{r^{n+1} - 1}{r - 1} + r^{n+1}$$

Common denominator:

$$\begin{aligned} r^0 + \dots + r^n + r^{n+1} &= \frac{r^{n+1} - 1}{r - 1} + \frac{(r - 1)r^{n+1}}{r - 1} \\ &= \frac{\cancel{r^{n+1}} - 1 + r^{n+2} - \cancel{r^{n+1}}}{r - 1} \\ &= \frac{r^{n+2} - 1}{r - 1}, \end{aligned}$$

which is exactly $P(n+1)$.

Thus, we have shown that $P(n) \rightarrow P(n+1)$, completing the proof.

8. [12 points] Consider the set S which is defined recursively as follows:

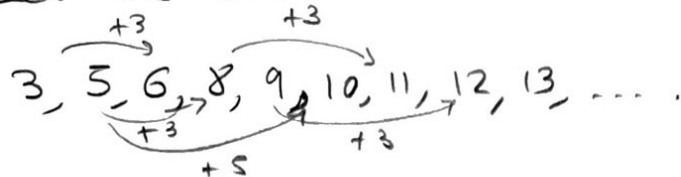
$$3 \in S \text{ and } 5 \in S$$

If $x \in S$ and $y \in S$, then $x + y \in S$.

(a) Is 7 in S ? Is 11 in S ?

(b) Find all the positive integers that are **not** in S .

Let us list the elements in S :



(a) Thus, $7 \notin S$, but $11 \in S$.

(b) 1, 2, 4, 7.

9. [12 points] Consider the statement

$P(n)$: "every amount of postage of 20 cents or more can be formed using just 5-cent and 6-cent stamps".

Prove this proposition using strong induction by completing the BASIC STEP outlined below and doing the INDUCTIVE STEP.

- BASIC STEP: $P(20)$, $P(21)$, $P(22)$, $P(23)$, and $P(24)$ are true. [You need to prove each base case.]

$P(20)$: use 5, 5, 5, 5

$P(21)$: use 5, 5, 5, 6

$P(22)$: use 5, 5, 6, 6

$P(23)$: use 5, 6, 6, 6

$P(24)$: use 6, 6, 6, 6.

- INDUCTIVE STEP: (do this step completely)

For $n \leq 24$, we know $P(n)$ is true.

Suppose that $n \geq 24$, and suppose that, for some n , $P(k)$ is true for all $k \leq n$.

This means that postage of k cents can be made using 5 and 6 cent stamps, as long as $k \leq n$.

To prove that $P(n+1)$ is true, we know that $P(n-4)$ is true, because $20 \leq n-4 \leq n$.

Therefore, we can make postage of $n-4$. To make postage of $n+1$ cents, just add one 5 cent stamp to the $n-4$ postage.