

CSI 35: Discrete Mathematics II. Midterm 1

Professor Luis Fernández

NAME: SOLUTION

INSTRUCTIONS:

- This exam contains 9 questions. 6 pages for a total of 104 points (points over 100 count as extra credit).
- You have 110 minutes to complete the exam.
- **You must show all your work** in order to get credit.
- You can use a non-graphing scientific calculator. No other electronic devices, notes or books are permitted.

1. [10 points] We recursively define a function f from the positive integers to the integers as follows: $f(1) = 2$ and $f(n) = -3 \cdot f(n-1) + 8$.

Evaluate the following.

(a) $f(1) = 2$

(given)

(b) $f(2)$

$$\begin{aligned} f(2) &= -3 \cdot f(1) + 8 \\ &= -3 \cdot (2) + 8 \\ &= -6 + 8 \\ &= \boxed{2} \end{aligned}$$

(c) $f(3)$

$$\begin{aligned} f(3) &= -3 \cdot f(2) + 8 \\ &= -3 \cdot (2) + 8 \\ &= \boxed{2} \end{aligned}$$

2. [12 points] Let $F(n) = -5n$, defined on integers $n \geq 1$. Find a *recursive* definition for F .

Definition of $F(n)$:

$$F(1) = -5$$

$$F(n) = F(n-1) + (-5), \text{ if } n \geq 2$$

3. [12 points] Consider the following program:

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if  $x < 0$  :  
     $x = x + 7$   
else:  
     $x = 6x$ 
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Suppose you are given that x is initialized to some positive integer. Show that after running the program, the value of x is even.

Suppose $x > 0$.

Then program skips the "if $x < 0$ " and jumps to the "else".

Therefore, we get: (new x) = $6 \cdot (\text{old } x)$

Whatever the initial value of x is, the new value will be $6x$, which is an even number, as $6x = 2(3x)$.

4. [12 points] Give a recursive definition of the set of all positive integers that are multiples of 7.

Let S be defined by

$$7 \in S$$

if $x \in S$ and $y \in S$, then $x+y \in S$.

Then S is the set of all positive integers that are multiples of 7.

5. [10 points] Consider the proposition: 5 divides $n^5 - n$ for $n \geq 2$. Suppose that we want to prove this proposition by induction.

(a) Carefully state the propositional function $P(n)$ that has to be proved.

$P(n)$ is: 5 divides $n^5 - n$

(b) Carefully state and prove the BASIC STEP.

Basic step: $P(2)$: 5 divides $2^5 - 2 = 30$
which is true: $5 \cdot 6 = 30$.

(c) Carefully state the INDUCTIVE STEP. Do not prove it.

If 5 divides $n^5 - n$ for some n ,
then 5 divides $(n+1)^5 - (n+1)$.

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6. [12 points] Write a recursive algorithm called " $a(n)$ ", for n a nonnegative integer, that computes the n^{th} term of the sequence a_n defined by $a_0 = 1$, $a_1 = 2$, and $a_n = a_{n-1} \cdot a_{n-2}$. You should write it as follows:

procedure $a(n$: n nonnegative integer)

if $n = 0$ return 1

else if $n = 1$ return 2

else .. return $a(n-1) \cdot a(n-2)$

7. [12 points] Let r be a real number. Prove by induction that $r^0 + r^1 + r^2 + \dots + r^n = \frac{r^{n+1} - 1}{r - 1}$.
Make sure to clearly state $P(n)$ and clearly explain the base and inductive steps.

$P(n)$ is: $r^0 + r^1 + \dots + r^n = \frac{r^{n+1} - 1}{r - 1}$

BASIC STEP

$P(0)$ is: $r^0 = \frac{r^{0+1} - 1}{r - 1}$

$1 = \frac{r - 1}{r - 1} = 1 \quad \checkmark \quad (P(0) \text{ is true})$

INDUCTIVE STEP for some n

Suppose that $P(n)$ is true, that is,

$r^0 + r^1 + \dots + r^n = \frac{r^{n+1} - 1}{r - 1}$ is true

- Add r^{n+1} to both sides to get

$r^0 + r^1 + \dots + r^n + r^{n+1} = \frac{r^{n+1} - 1}{r - 1} + r^{n+1}$

- Common denominator:

$$\begin{aligned} r^0 + r^1 + \dots + r^{n+1} &= \frac{r^{n+1} - 1}{r - 1} + \frac{r^{n+1}(r - 1)}{r - 1} \\ &= \frac{r^{n+1} - 1 + r^{n+2} - r^{n+1}}{r - 1} \\ &= \frac{r^{n+2} - 1}{r - 1} \end{aligned}$$

This is exactly $P(n+1)$.

Thus, we proved that if $P(n)$ is true, then $P(n+1)$ is true, completing the proof.

8. [12 points] Consider the set S which is defined recursively as follows:

$$2 \in S \text{ and } 5 \in S$$

If $x \in S$ and $y \in S$, then $x + y \in S$.

(a) Is 20 in S ? Is 13 in S ?

(b) Find all the positive integers that are **not** in S .

Let us list elements of S :

2, 4, 5, $\overset{+2}{\curvearrowright}$ 6, 7, 8, 9, 10, 11, ... etc.

(a) Both 13 and 20 are in S

(b) 1 and 3. All the other positive integers are in S .

9. [12 points] Consider the statement

$P(n)$: "every amount of postage of 20 cents or more can be formed using just 5-cent and 6-cent stamps".

Prove this proposition using strong induction by completing the BASIC STEP outlined below and doing the INDUCTIVE STEP.

- BASIC STEP: $P(20)$, $P(21)$, $P(22)$, $P(23)$, and $P(24)$ are true. [You need to prove each base case.]

$P(20)$: use 5, 5, 5, 5

$P(21)$: use 5, 5, 5, 6

$P(22)$: use 5, 5, 6, 6

$P(23)$: use 5, 6, 6, 6

$P(24)$: use 6, 6, 6, 6.

- INDUCTIVE STEP: (do this step completely)

For $n \leq 24$, we know $P(n)$ is true.

Suppose that $n \geq 24$, and suppose that, for some n , $P(k)$ is true for all $k \leq n$.

This means that postage of k cents can be made using 5 and 6 cent stamps, as long as $k \leq n$.

To prove that $P(n+1)$ is true, we know that $P(n-4)$ is true, because $20 \leq n-4 \leq n$.

Therefore, we can make postage of $n-4$. To make postage of $n+1$ cents, just add one 5 cent stamp to the $n-4$ postage.