# CSI 35: Discrete Mathematics II. Midterm 1

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NAME: SOLUTION

**INSTRUCTIONS:**

- This exam contains 9 questions, 5 pages for a total of 106 points (points over 100 count as extra credit).
- You have 110 minutes to complete the exam.
- **You must show all your work** in order to get credit.
- You can use a non-graphing scientific calculator. No other electronic devices, notes or books are permitted.

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1. [12 points] We recursively define $f$ as follows: $f(1) = 2$ and $f(n) = 5 \cdot f(n-1) + 10$.

   Evaluate the following.

   1. $f(1) = \boxed{2}$
   2. $f(2) = 5 \cdot f(1) + 10$
      $= 10 + 10$
      $= \boxed{20}$
   3. $f(3) = 5 \cdot f(2) + 10$
      $= 5 \cdot 20 + 10$
      $= \boxed{110}$

2. [12 points] Let $F(n) = 5n$, defined on integers $n \geq 1$. Find a recursive definition for $F$.

   F(n):
   If n=1, return 5
   else return 5+F(n-1)

3. [12 points] Consider the following program:

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if x < 0 :
    x = x + 7
else:
    x = 6x
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Suppose you are given that $x$ is initialized to some positive integer. Show that after running the program, the value of $x$ is even.

Since initial x is positive, the program jumps to the "else" part and returns $6x$, which equals $2 \cdot (3x)$, which is even.

4. [12 points] Give a recursive definition of the set of all positive integers that are multiples of 7.

Set S:

$7 \in S$

If $x, y \in S$, then $x+y \in S$.

5. [10 points] Consider the proposition: 5 divides $n^5 - n$ for $n \geq 2$.

In order to prove this by induction you should (1) State what statement $P(n)$ you are doing induction on, (2) Do the base step: show $P(n)$ is true for the base case, and (3) Do the inductive step.

Just do (1) and (2). That is, state exactly what $P(n)$ is, then state and prove the base case. Do **not** do the inductive step.

$P(n)$: "5 divides $n^5 - n$".

$P(2)$ is true:

$P(2)$ is "5 divides $2^5 - 2$".

$2^5 - 2 = 32 - 2 = 30$, and 5 divides 30,

so $P(2)$ is true.

6. [12 points] Prove by induction that $2 + 4 + 6 + \cdots + 2n = n(n+1)$.

(That is, the sum of the first $n$ positive even integers is $n(n+1)$.)

$P(n)$: ~~1+~~ $2 + 4 + \ldots + 2n = n(n+1)$.

Basis step: $P(1)$ is true.

$P(1)$ says "$2 = 1 \cdot (1+1)$", which is true:

$2 = 2$ ✓.

Inductive step:

Suppose $P(n)$ true. Thus, $2 + \ldots + 2n = n(n+1)$.

Add $2(n+1)$ to both sides:

$2 + \ldots + 2n + 2(n+1) = n(n+1) + 2(n+1) = (n+1)(n+2)$

$\Rightarrow 2 + \ldots + 2(n+1) = (n+1)(n+2)$, which is $P(n+1)$.

7. [12 points] Devise a recursive algorithm called "$a(n)$", for $n$ is a nonnegative integer, that computes the $n^{\text{th}}$ term of the sequence $a_n$ defined by $a_0 = 1$, $a_1 = 2$, and $a_n = a_{n-1} \cdot a_{n-2}$.

a(n):

If n = 0, return 1

else if n = 1, return 2

else return a(n-1) · a(n-2).

8. [12 points] Consider the set $S$ which is defined recursively as follows:

$2 \in S$ and $5 \in S$
If $x \in S$ and $y \in S$, then $x + y \in S$.

1. Is 20 in $S$? Is 13 in $S$?
2. Find all the positive integers that are **not** in $S$.

1) 20 is in S because:
$5 \in S$, so $5+5=10 \in S$, so $5+10=15 \in S$,
so $5+15=\underline{\underline{20 \in S}}$.

$13 \in S$ because:
$5 \in S$ and $2 \in S$, so $5+2=7 \in S$, so $7+2=9 \in S$,
so $9+2=11 \in S$, so $11+2=\underline{\underline{13 \in S}}$.

2) $1 \notin S$.
$3 \notin S$.
$2 \in S$ and $2+2+\dots+2 \in S$, so every even number is in S.
$5 \in S$, so $7=5+2 \in S$, and $5+2+\dots+2 \in S$, so every odd number $\geq 5$ is in S.
Thus, the only positive integers not in S are $\underline{1 \text{ and } 3}$.

9. [12 points] Prove that every amount of postage of 6 cents or more can be formed using just 3-cent and 4-cent stamps. Use strong induction with the proof starting as follows:

- Let $P(n)$ be "postage of $n$ cents can be formed using just 3-cent and 4-cent stamps."
- Base cases: $P(6), P(7)$, and $P(8)$. [You need to state and prove each base case]

Now finish the proof by doing the inductive step.

Base cases:

P(6) is "6 cents can be formed using 3 & 4 cents"

True, since $\underline{6 = 3+3}$.

P(7) = "7 cents can be formed using 3 & 4 cents"

True since $\underline{7 = 3+4}$

P(8) = "8 cents can be formed using 3 & 4 cents"

True since $\underline{8 = 4+4}$

<u>Inductive step</u>

Assume that for all $k \le n$, $P(k)$ is true, so "k cents can be formed using 3 & 4 cents" is true for every $k \le n$.

To prove that we can form $n+1$, note that, by the induction hypothesis, we can form $n-2$, since $n-2 \le n$.

So, to form $n+1$, form $n-2$ first, and then add a 3 cent stamp. This forms $n+1$, so $P(n+1)$ is true.