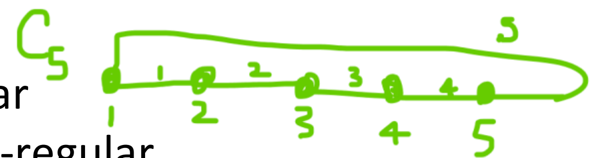
CSI 35 D02 — BCC, CUNY

Wed 11/5/25 — E. Antonakos for L. Fernandez

Previously, From **10.1/10.2**

SIMPLE GRAPH: V non-empty, undirected, no loops, no parallel edges,
MULTIGRAPH: Simple graph with the possible addition of parallel edges
PSEUDOGRAPH: multipgrpah with the possible addition of loops
DIRECTED GRAPH: edges are *ordered* pairs of vertices, loops allowed but no parallel edges

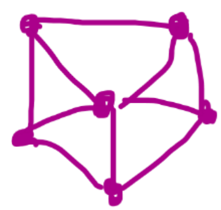


Cycle C_n $|V| = n,\ |E| = n$ 2-regular
Complete K_n $|V| = n,\ |E| = ?$ $(n-1)$-regular
so total degree is $n(n-1),\ |E| = \frac{n(n-1)}{2}$
(hyper) Cube Q_n $|V| = 2^n,\ |E| = ?$ n-regular
so total degree is $2^n \cdot n,\ |E| = \frac{2^n \cdot n}{2} = n \cdot 2^{n-1}$

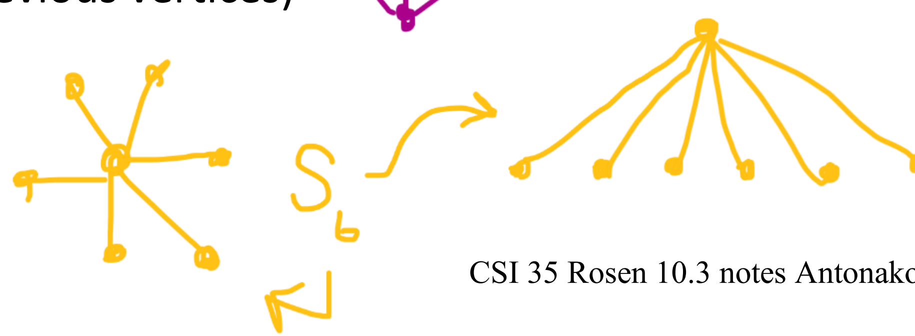
complete Bipartite $K_{n,m}$

New:

Wheel W_n $|V| = n + 1,\ |E| = 2n$
Add a vertex to C_n of degree n (connect to all previous vertices)



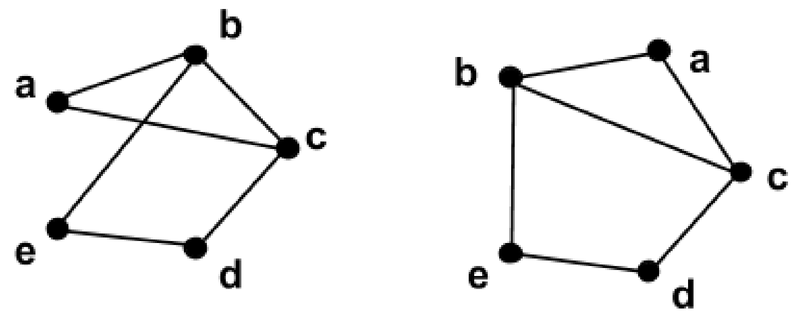
Star S_n $|V| = n + 1,\ |E| = n$
one vertex of deg n, the rest are deg 1.



Rosen 10.3 Graphs Representations (& Graph Isomorphisms)

The way a graph is drawn on a 2-dimensional surface (paper or screen) is not part of the graph itself.

Figure 1: Two drawings of an undirected graph.



The two graphs look different, but that is only because they are drawn differently. The two graphs are actually the same graph because they have the same vertex and edge sets

There are tWo common ways to represent graphs for use in algorithms that make the information about the graph more accessible:
adjacency list representation for graphs and ***matrix representation*** for graphs.

Adjacency list representation for graphs

In the adjacency list representation of a graph, each vertex has a list of all its neighbors. If graph is undirected, if vertex a is in b's list of neighbors, then b must also be in a's list of neighbors.

If a graph is represented using adjacency lists, the time required to list the neighbors of a vertex v is proportional to deg(v), the number of vertices to be listed. In order to determine if $\{a, b\}$ is an edge, it is necessary to scan the list of a's neighbors or the list of b's neighbors.
In the worst case, the time required is proportional to the larger of deg(a) or deg(b).

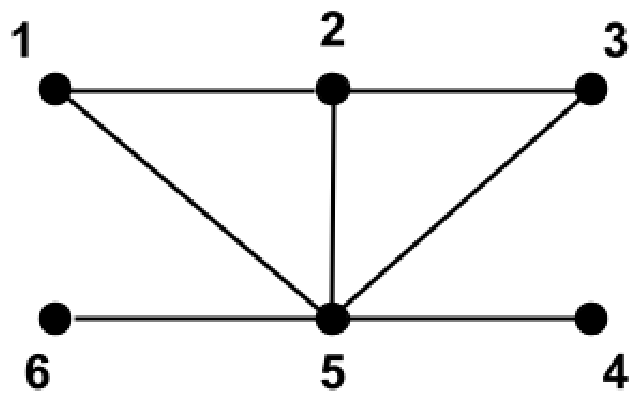
Matrix representation for graphs

The matrix representation for a graph with n vertices is an $n \times n$ matrix whose entries are all either 0 or 1, indicating whether or not each edge is present. If the matrix is labeled M, then $M_{i,j}$ denotes the entry in row i and column j. For a matrix representation, the vertices of the graph are labeled with integers in the range from 1 to n. Entry $M_{i,j} = 1$ if and only if $\{i, j\}$ is an edge in the graph. Since the graph is undirected, $M_{i,j} = M_{j,i}$ because $\{i, j\}$ and $\{j, i\}$ refer to the same edge which is either present in the graph or not. Thus the matrix representation of an undirected graph is symmetric about the diagonal, meaning that it is a mirror image of itself along the diagonal extending from the upper left corner to the lower right corner.

If a graph is represented with a matrix, determining if $\{i, j\}$ is an edge only requires examining matrix element $M_{i,j}$ which can be done in O(1) time. In order to list the neighbors of a vertex j, it is necessary to scan all of row j. Each 1 encountered corresponds to a neighbor of vertex j. The time to required to list the neighbors of vertex j requires Ω(n) time. SEE END OF NOTES**

Exercise 1: From a graph drawing to other graph representations.

A graph G is depicted in the diagram below.

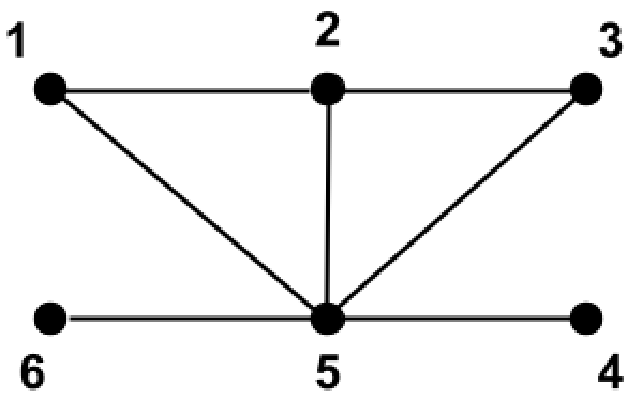


(a) Give the adjacency list representation of G.

(b) Give the matrix representation of G.

Exercise 1: From a graph drawing to other graph representations.

A graph G is depicted in the diagram below.



From 10.1 we would define $G = (V, E)$ as follows:

$V = \{1,2,3,4,5,6\}$

$E = \{\{1,2\}, \{1,5\}, \{2,3\}, \{2,5\}, \{3,5\}, \{4,5\}, \{5,6\}\}$

(a) Give the adjacency list representation of G.

Vertex→adjacent vertices

1→2, 5

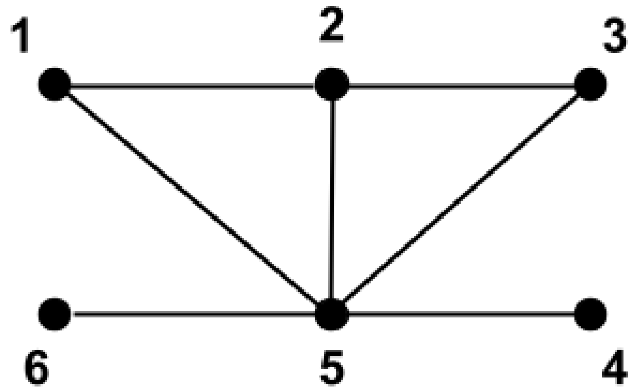
2→1,3,5

3→2,5

4→5

5→1,2,3,4,6

6→5

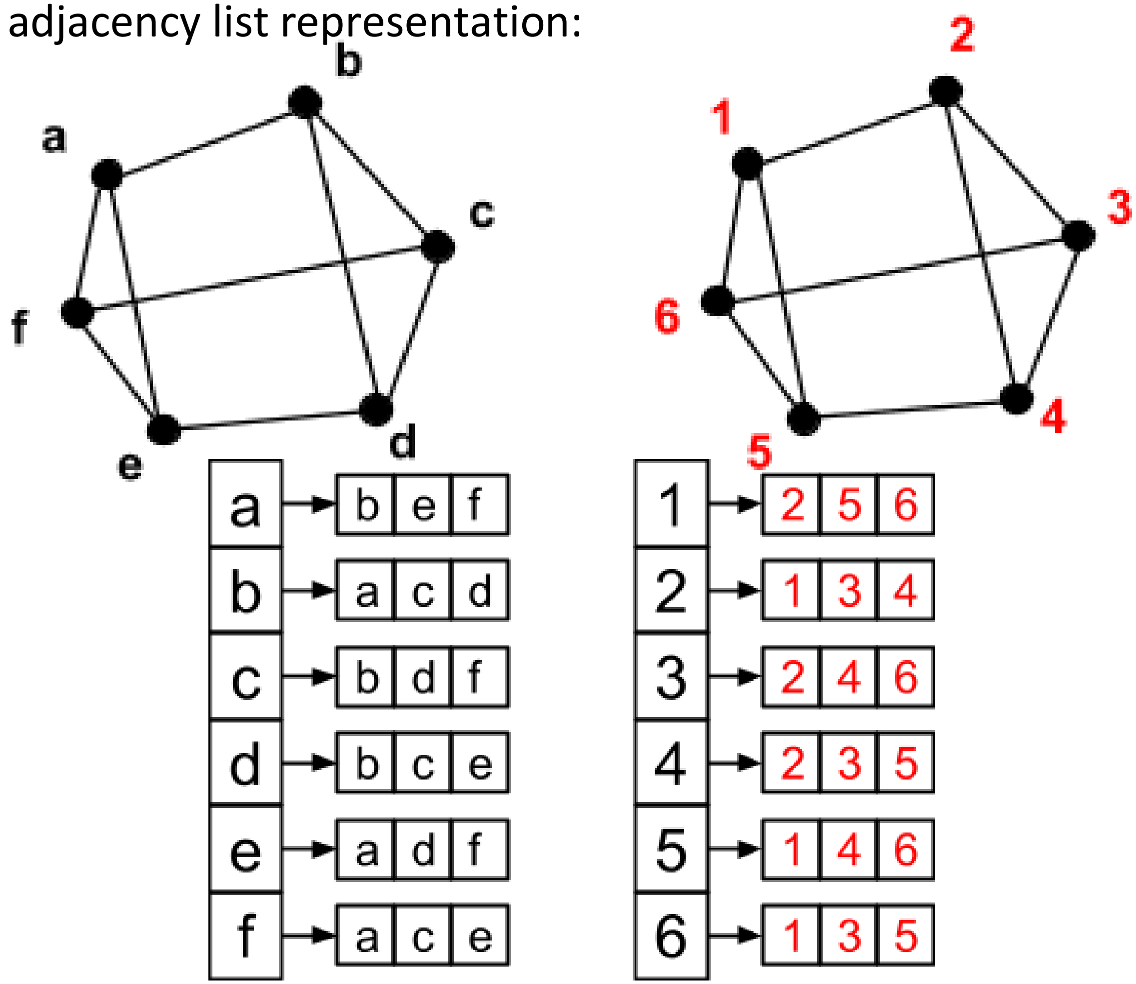


(b) Give the matrix representation of G.

	1	2	3	4	5	6
1	0	1	0	0	1	0
2	1	0	1	0	1	0
3	0	1	0	0	1	0
4	0	0	0	0	1	0
5	1	1	1	1	0	1
6	0	0	0	0	1	0

10.3 (Representing Graphs &) **Graph Isomorphisms**

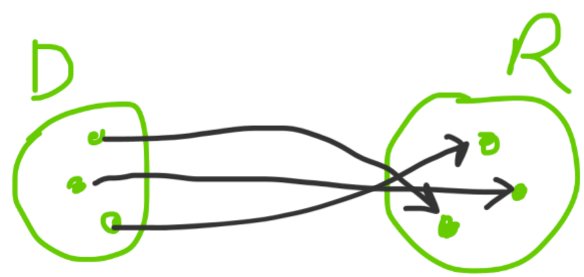
Two graphs are said to be **isomorphic** if there is a correspondence between the vertex sets of each graph such that there is an edge between two vertices of one graph if and only if there is an edge between the corresponding vertices of the second graph. The graphs are not identical but the vertices can be relabeled so that they are identical. This can be easier to see in a drawing than in an adjacency list representation:



Definition: **Isomorphic graphs**.
Let $G = (V, E)$ and $G' = (V', E')$.
Graphs G and G' are **isomorphic** if there is a bijection $f: V \rightarrow V'$ such that for every pair of vertices $x, y \in V, \{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$.
The function f is called an **isomorphism** from G to G'.

Bijection $f: D \rightarrow R$ is a function that is *one-to-one* and *onto*

One-to-one (injective):
each range element is the image of only one domain element
If $f(x) = f(y)$, then $x = y$.



Onto (surjective):
The Co-Domain = Range

A property is said to be **preserved under isomorphism** if whenever two graphs are isomorphic, one graph has the property if and only if the other graph also has the property.

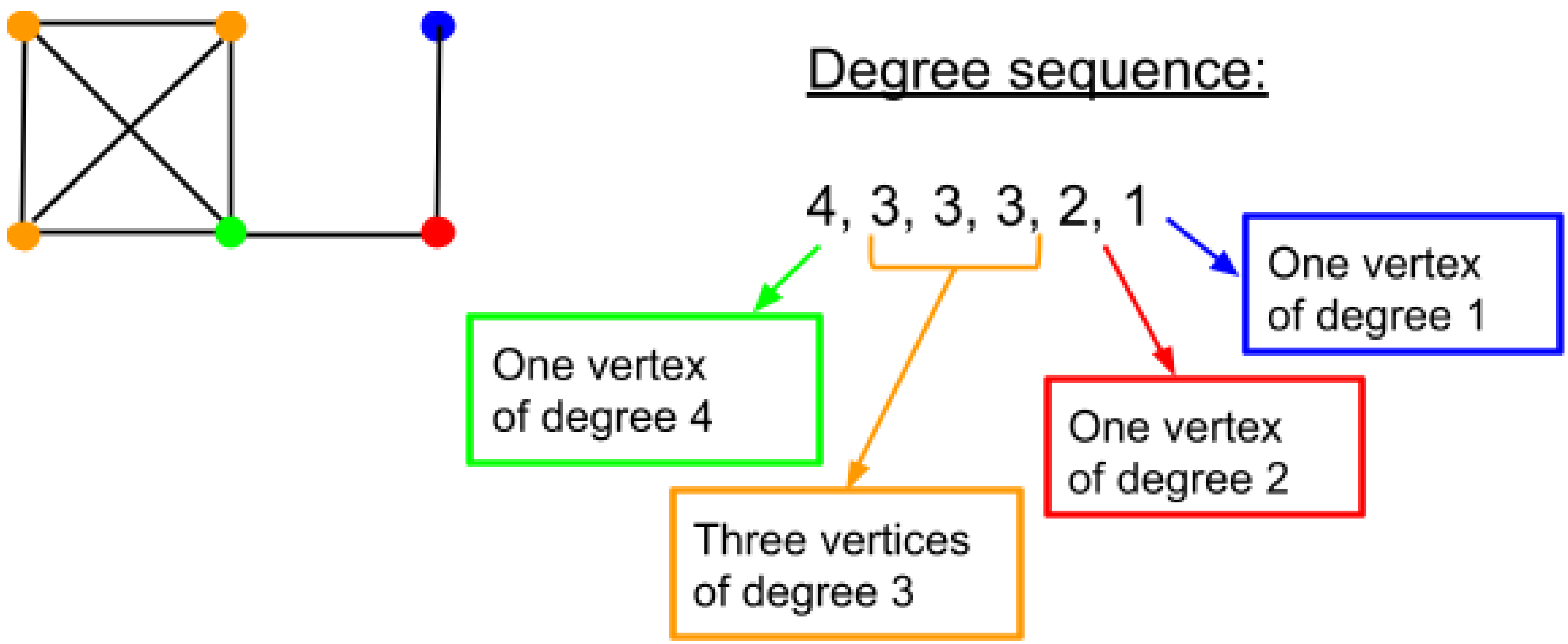
Theorem 1: **Vertex degree preserved under isomorphism**.
Consider two graphs, G and G'. Let f be an isomorphism from G to G'.
For each vertex v in G, the degree of v in G equals the degree of vertex $f(v)$ in G'.

Proof.

Let $N(v)$ denote the set of neighbors of vertex v in G. The $\deg(v) = |N(v)|$. A vertex $w \in N(v)$ if and only if $\{v, w\}$ is an edge in G. By the definition of isomorphism, $\{v, w\}$ is an edge in G if and only if $\{f(v), f(w)\}$ is an edge in G'. The neighbors of vertex $f(v)$ in G' is the set $\{f(w): w \in N(v)\}$. Since f is an isomorphism, f is one-to-one. Therefore, the cardinality of $\{f(w): w \in N(v)\}$ is the same as the cardinality of $N(v)$. Thus, the degree of v in G is the same as the degree of $f(v)$ in G'. ∎

Definition: The **degree sequence** of a graph is a list of the degrees of all of the vertices in non-increasing order.

Figure 2: The degree sequence of a graph.

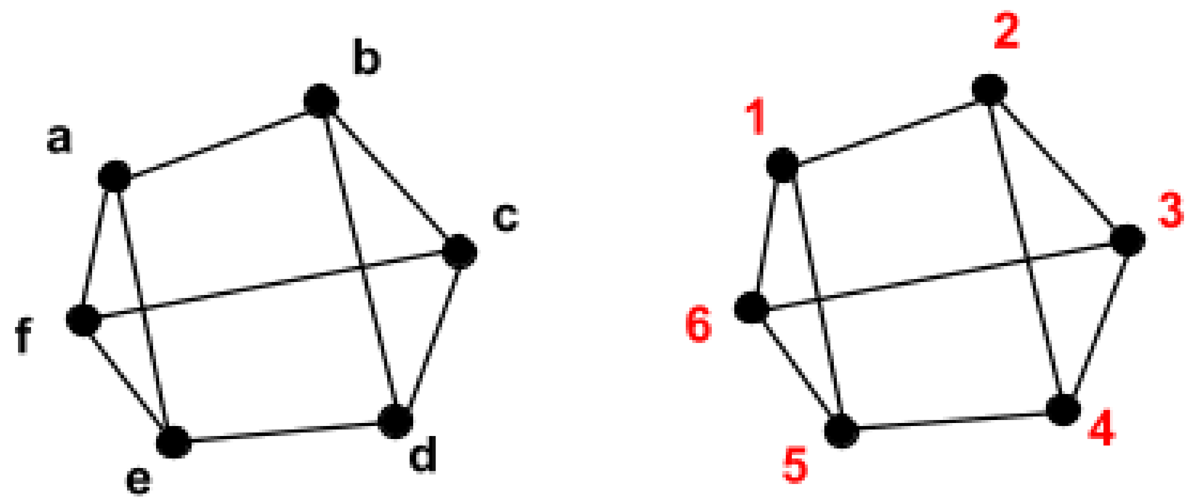


Theorem 2: **Degree sequence preserved under isomorphism**.
The degree sequence of a graph is preserved under isomorphism.

Proof. (skipped)

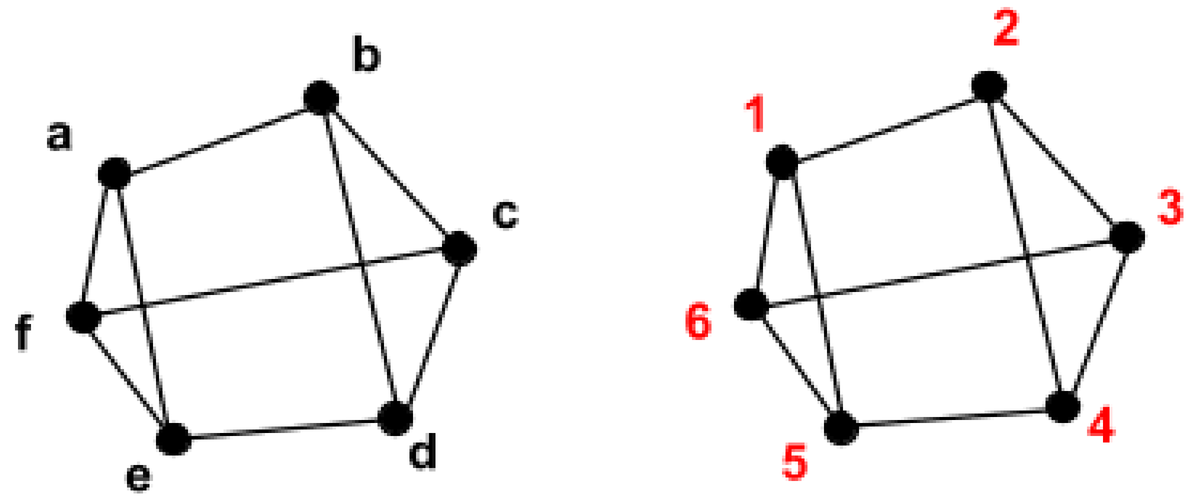
Suppose that there is an isomorphism f from graph G to G'. We will show that the degree sequence of G is the same as the degree sequence of G'.

Let the degree sequence of G be $d_1, d_2, \ldots, d_n$.
The degree sequence is, by definition, non-increasing, so $d_1 \geq d_2 \geq \cdots \geq d_n$.
Order the vertices of G by degree so that vertex v_j has degree d_j. For each vertex v_j in G, the degree of v_j in G is equal to the degree of vertex $f(v_j)$ in G'. Therefore, if the degrees of each of the vertices $f(v_1), \ldots, f(v_n)$ are listed in order, that corresponds to a non-increasing degree sequence which is identical to $d_1, d_2, \ldots, d_n$. ■



Exercise 2:

What is the isomorphism between these two graphs (G on left and G' on right)?



Exercise 2:

What is the isomorphism between these two graphs (G on left and G' on right)?

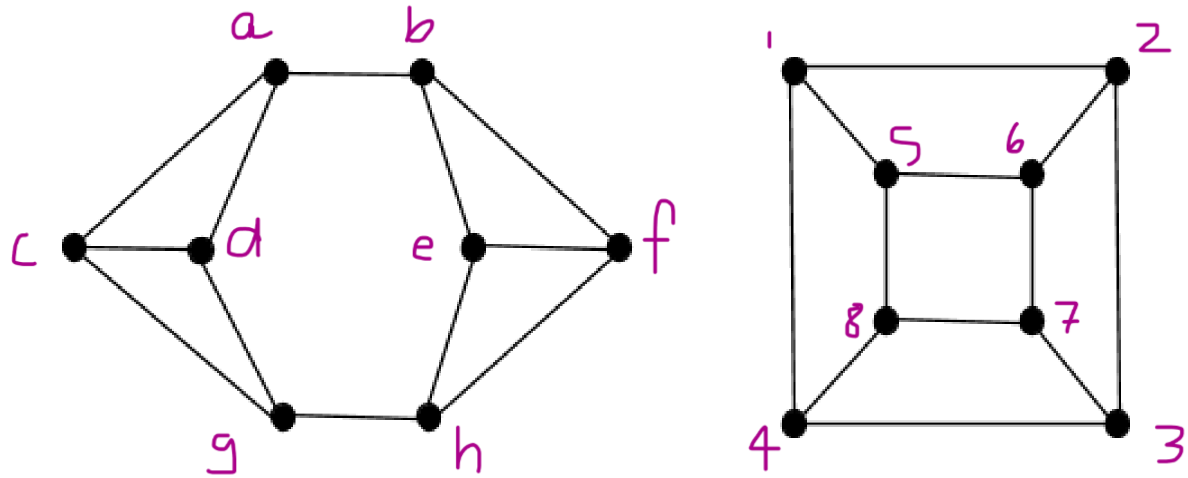
$$\boldsymbol{f}: V \rightarrow V' = \{(a, 1), (f, 6), (e, 5), (b, 2), (c, 3), (d, 4)\}$$

What's the degree sequence here? Easy! It's a 3-regular graph: 3,3,3,3,3,3

Let's say this is due to ordering vertices of V like this a, b, c, f, e, d.
The ordering from the image of $\boldsymbol{f}$ is: 1,2,3,6,5,4 which also has the degree sequence 3,3,3,3,3.

Note: Isomorphic graphs have the same degree sequence but just because two graphs have the same degree sequence doesn't mean that they are isomorphic. See below:

<u>Figure 3</u>: Two non-isomorphic 3-regular graphs. (See also notes from 10.1)



Why not isomorphic? Look at some subgraphs. For example, C_3 is a subgraph on the left but not on the right.

Try to build the isomorphism $\boldsymbol{f}$:

What if $\boldsymbol{f}(a) = 1$? Then the neighbors of a (b, c, and d) must map to the neighbors of 1 (2, 5, and 4). But there are no edges between 2, 5, and 4 while there is between c and d so the neighbors of a can't map to those of 1. Similarly, $\boldsymbol{f}(a)$ can't be 2 or 3 or 4 or … or 8.

Graph theory is concerned with properties of graphs that *are* preserved under isomorphism. Properties preserved under isomorphism relate to the structure of graphs.

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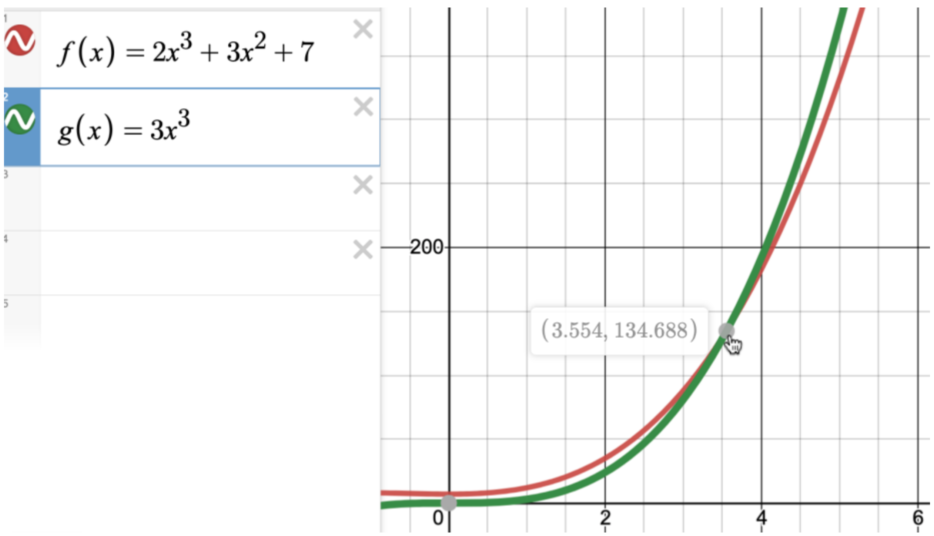
Section (check #?) (Asymptic grown functions) for details on growth functions O(n) and Ω(n).

The notation f = O(g) is read "f is Oh of g".
f = O(g) essentially means that the function f(n) is less than or equal to g(n), if constant factors are omitted and small values for n are ignored. In the expressions $7n^3$ and $5n^2$, the 7 and the 5 are called constant factors because the values of 7 and 5 do not depend on the variable n.
If f is O(g), then there is a positive number c such that when f(n) and c·g(n) are graphed, the graph of c·g(n) will eventually cross f(n) and remain higher than f(n) as n gets large.

Consider the function $f(n) = 2n^3 + 3n^2 + 7$. There are many functions g(n) such that f = O(g). For example, f(n) is $O(4n^3 + 2n + 1)$. However, the idea is to select a function g that characterizes the asymptotic growth of f as simply as possible. Therefore, g is selected so as to eliminate all unnecessary constants and additional terms. Instead of saying that $f = O(4n^3 + 2n + 1)$, one would typically say that $f = O(n^3)$.



Definition: Oh notation.

Let f and g be two functions from $\mathbf{Z}^+$ to $\mathbf{R}^+$. Then f = O(g) if there are positive real numbers c and n_0 such that for any $n \in \mathbf{Z}^+$ such that $n \geq n_0$,

$$f(n) \leq c \cdot g(n).$$

The O notation serves as a rough *upper* bound for functions (disregarding constants and small input values). The Ω notation is similar, except that it provides a *lower* bound on the growth of a function:

Definition: Ω Notation.

Let f and g be two functions from $\mathbf{Z}^+$ to $\mathbf{R}^+$. Then f = Ω(g) if

there are positive real numbers c and n_0 such that for any $n \geq n_0$,

$$f(n) \geq c \cdot g(n).$$

The notation f = Ω(g) is read "f is **Omega** of g".