

10.1 Introduction to Graphs

Directed graphs were introduced in the context of relations. Here we are concerned with undirected graphs. In an **undirected graph**, the edges are unordered pairs of vertices, which is useful for modeling relationships that are symmetric.

For example, an undirected graph could be used to model sibling relationships within a family. Unlike parent/child relationships in which the two people have different roles, sibling relationships are symmetric. Two people are mutual siblings or neither one is the sibling of the other.

A **graph** consists of a **pair of sets (V, E)**: **V** is a set of **vertices** & **E** is a set of **edges**.

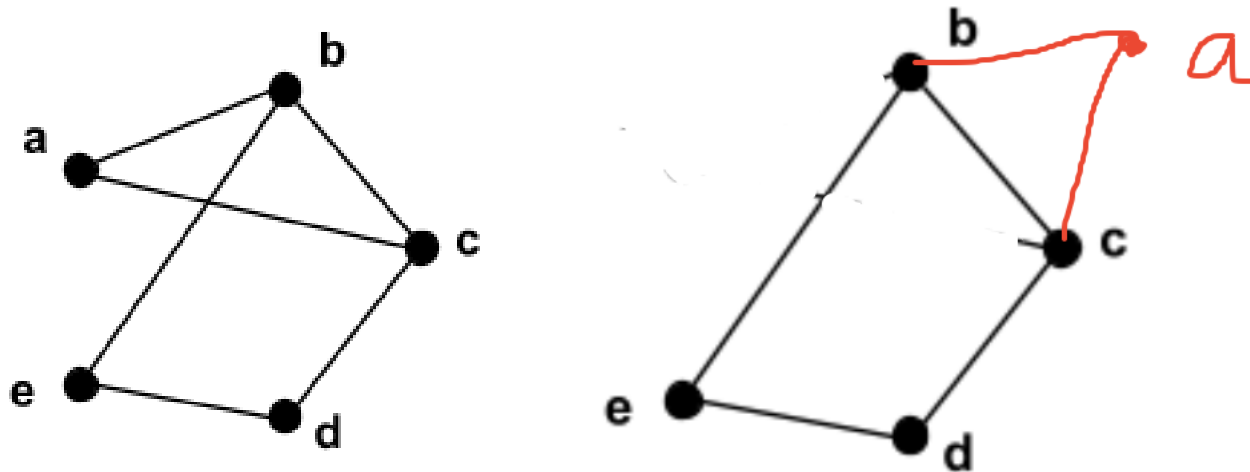
A vertex $v \in V$ is usually represented pictorially by a dot with a label.

An edge $e = \{a, b\} \in E$ is a set of two vertices $\{a, b\}$ and is drawn as a line connecting the two vertices.

A graph is **finite** if the vertex set is finite.

This material will only be concerned with *finite* graphs.

Fig 1: An Undirected Graph (V, E) – two versions of the **same** graph



$$V = \{a, b, c, d, e\}$$

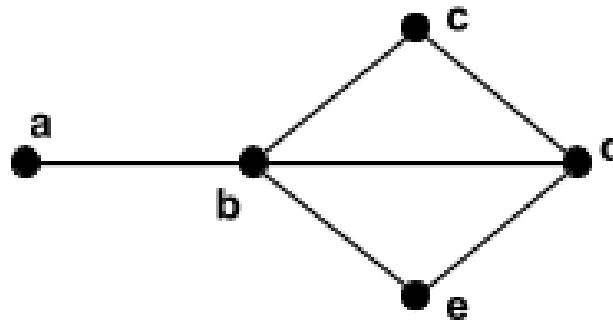
$$E = \{\{a, b\}, \{a, c\}, \{b, c\}, \{b, e\}, \{c, d\}, \{d, e\}\}$$

Terminology

- **parallel** edges edges connecting the same pair of vertices
- **self-loop** edge For $G = \{V, E\}$, there is $\{v, v\} \in E$ for some $v \in V$
- **simple** graph no loops, no parallel edges
- **adjacent** vertices are connected by an edge
- **neighbor** vertex same as adjacent
- **endpoint** vertex one end of an edge
- **incident** edge edge in/out of particular vertex
- in a simple graph: **degree** of a vertex b , denoted $\deg(b)$ is the number of edges incident to b
- **total degree** of a graph : sum of its vertices' degrees
- **regular** graph, **d-regular** graph: all vertices have the same degree (**d**)
- **subgraph** of $G' = (V', E')$ is a graph where $G = (V, E)$ is a graph and $V' \subseteq V$ and $E' \subseteq E$

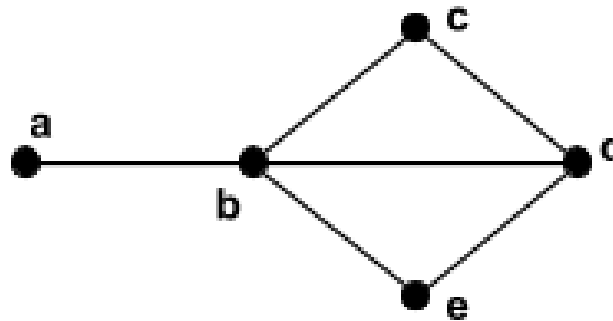
(You can erase any subset of edges but if you erase a vertex, you must erase all its incident edges. Disconnected graphs are okay.)

Exercise 10.1: What is the total degree of this graph?



Answer: $\sum_{v \in V} \deg(v) =$

Exercise 10.1: What is the total degree of this graph?



Answer: $\sum_{v \in V} \deg(v) = \deg(a) + \deg(b) + \deg(c) + \deg(d) + \deg(e) =$
 $1 + 4 + 2 + 3 + 2 = 12$

Each edge is counted twice in the total degree, once for each endpoint so here
 $2(6 \text{ edges}) = 12$

Theorem 10.1: Number of edges is half the total degree.

Let $G = (V, E)$ be an undirected graph.

Then twice the number of edges is equal to the total degree:

$$\sum_{v \in V} \deg(v) = 2 \cdot |E| .$$

Prove this.

Theorem 10.1: Number of edges is half the total degree.

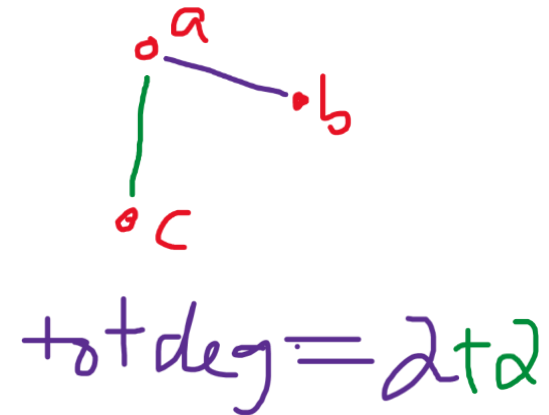
Let $G = (V, E)$ be an undirected graph. Then twice the number of edges is equal to the total degree:

$$\sum_{v \in V} \deg(v) = 2 \cdot |E|.$$

PROOF: By induction on number of edges $k = |E|$ (cardinality of E)

Base case: $n = 0$

$$0 = \sum_{v \in V} \deg(v) = 2 \cdot 0 = 0$$



Since the $\deg(v)=0$ for all $v \in V$

Assume that $\sum_{v \in V} \deg(v) = 2|E|$ holds for a graph with k edges. Show that it holds for a graph with $k+1$ edges

Graph with k edges -- and then add one more edge. (If you *can* add edge that's not a loop/parallel to another edge – see more formal presentation below.)

By IH $\sum_{v \in V} \deg(v) = 2k$ is true. Adding one edge adds 2 to the total degree. To keep it an equation, add 2 to right

$$2 + \sum_{v \in V} \deg(v) = 2k + 2 = 2(k + 1)$$

And edge count has indeed increased by 1

Proof (#2) of Theorem 10.1: The number of edges is half the total degree.

Proof. (Formal)

The proof is by induction on the number of edges in the graph.

Let $m = |E|$ be the number of edges in G .

Base case: $m = 0$. If there are no edges in the graph then the degree of every vertex is 0.

$$\sum_{v \in V} \deg(v) = 0 = 2m$$

Now assume that the theorem holds for all graphs m edges.

We need to show that the theorem holds with all graphs with $m + 1$ edges.

Consider a graph with $m + 1$ edges. Consider an edge $\{a, b\}$ in the graph. Remove the edge $\{a, b\}$ so that the graph now has only m edges. Let $\deg(v)$ denote the degree of vertex v after the edge $\{a, b\}$ has been removed. By induction, we know that the theorem holds for all graphs with m edges, so it holds after the edge $\{a, b\}$ is removed from the graph:

$$\sum_{v \in V} \deg(v) = 0 = 2m$$

When the edge $\{a, b\}$ is added back into G , the degree of vertex a becomes $\deg(a) + 1$ and the degree of vertex b becomes $\deg(b) + 1$. The degree of all other vertices in G remain unchanged.

Therefore, the new total degree is

$$2 + \sum_{v \in V} \deg(v) .$$

By the inductive hypothesis (IH),

$$2 + \sum_{v \in V} \deg(v) = 2 + 2m = 2(m + 1)$$

The first equality uses the inductive hypothesis (shown in red) by substituting $2m$ for $\sum_{v \in V} \deg(v)$. Therefore, after the edge $\{a, b\}$ is added back into the graph, the total degree is equal to $2(m + 1)$ which is twice the number of edges. ■

Exercise 10.2: Reasoning about regular graphs.

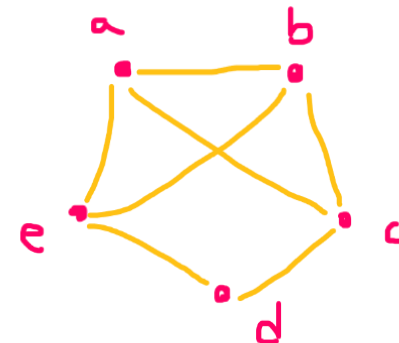
(a) Is it possible to have a 3-regular graph with five vertices? If such a graph is possible, draw an example. If such a graph is not possible, explain why not.

(b) Is it possible to have a 3-regular graph with six vertices? If such a graph is possible, draw an example. If such a graph is not possible, explain why not.

Exercise 10.2: Reasoning about regular graphs.

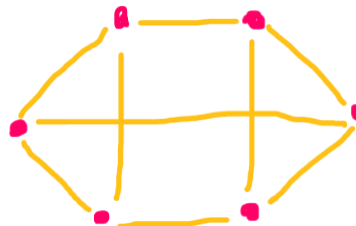
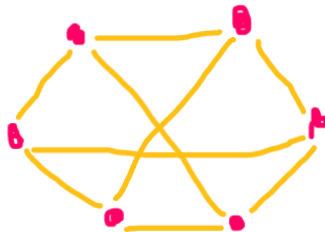
(a) Is it possible to have a 3-regular graph with five vertices? If such a graph is possible, draw an example. If such a graph is not possible, explain why not.

NO! See pic at right. Except for vertex d, the vertices have Degree = 3. But to make d degree 3 one of the others must become degree 4.

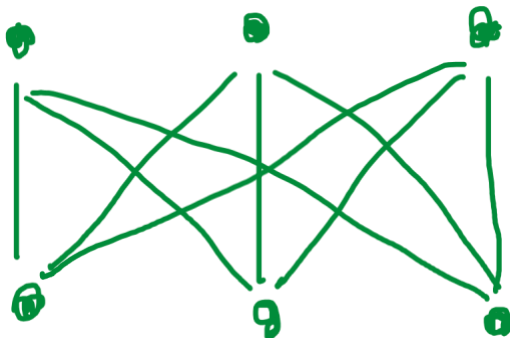


If 5 vertices each have degree 3 then the total degree of the graph is $(5)(3) = 15$
But the total degree of a graph must be even since $2|E|$. (see Thm above)

(b) Is it possible to have a 3-regular graph with six vertices? If such a graph is possible, draw an example. If such a graph is not possible, explain why not. Yes, several options



Here's an alternative:

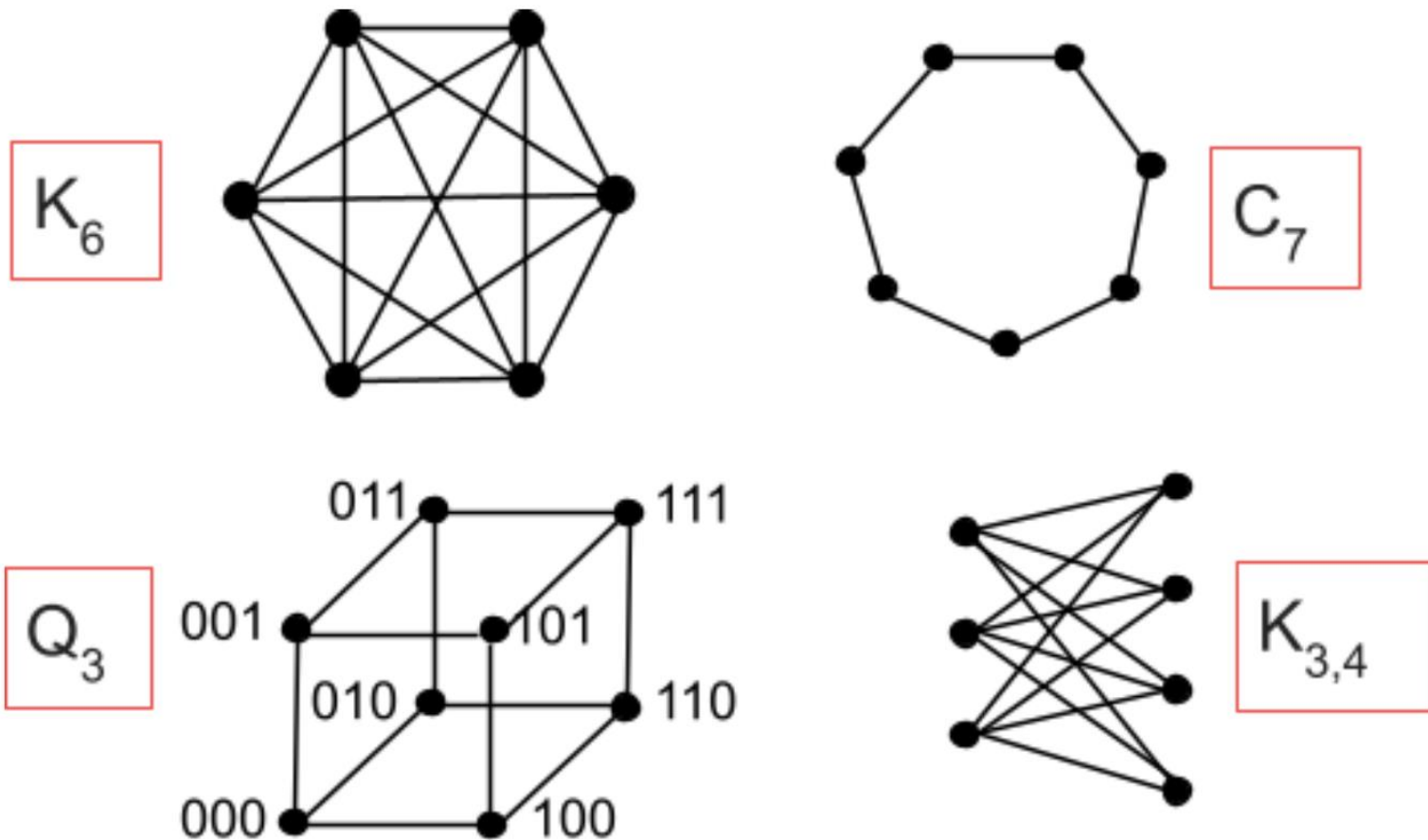


$\leftarrow 3, 3$

Examples: Undirected graphs in applications.

- Molecular graphs: vertices of the graph are the atoms in a molecule. There is an edge between two atoms if they form a bond. In molecular graphs, vertices are labeled with the type of atom and edges are labeled with the type of bond. The structure of a molecular graph reveals important information about the chemical properties of the molecule.
- Scheduling constraints: consider a school that has to schedule a set of courses for a given semester. The vertices of the graph represent the courses to be scheduled. There is an edge between two vertices if the corresponding courses have a conflict. For example, if the meeting times of the courses have already been determined, two courses that have overlapping times can not be scheduled in the same room. There would be an edge between any two courses whose meeting times overlap.
- Communication network: the vertices of a graph denote switches in a communication network. There is an edge between two switches if there is a two-way communication link between the two switches. Network designers would like to design a communication network such that even if a few communication links fail, it is still possible for every switch in the network to send a message to every other switch.
- Social network: consider a graph whose vertices represent individuals. There is an edge between two people if they are acquainted. (Assume that the property of being acquainted is mutual: if person A is acquainted with person B, then person B is acquainted with person A). Sociologists study social networks to understand how information spreads in communities or how societies evolve. The advent of online social networks has allowed social scientists to study social patterns on a much larger scale.

Figure 2: Common graphs in graph theory.



The following graphs are parameterized by a positive integer n .
In the case of $K_{n,m}$, n and m must both be positive integers.

- K_n is called the **complete graph** on n vertices. K_n has an edge between every pair of vertices. K_n is sometimes called a clique of size n or an n -clique.
- C_n is called a **cycle** on n vertices. The edges connect the vertices in a ring. Note that C_n is well defined only for $n \geq 3$.
- Q_n is the n -dimensional **hypercube**. It has 2^n vertices. Each vertex is labeled with an n -bit string. Two vertices are connected by an edge if their corresponding labels differ by only one bit. For example, in a 5-dimensional hypercube, the vertex labeled 11001 would have an edge to 11011 because the two strings only differ in the 4th location.
- $K_{n,m}$ has $n + m$ vertices. (**bipartite**) The vertices are divided into two sets: one with m vertices and one set with n vertices. There are no edges between vertices within a set, but there is an edge between every vertex in one set and every vertex in the other set.

Exercise 10.3: Reasoning about common graphs.

(a) How many edges are in $K_{3,4}$? Is $K_{3,4}$ a regular graph?

(b) How many edges are in K_5 ? Is K_5 a regular graph?

(c) What is the largest n such that $K_n = C_n$?

(d) For what value of n is $Q_2 = C_n$?

(e) Is Q_n a regular graph for $n \geq 1$?
If so, what is the degree of the vertices in Q_n ?

Exercise 10.3: Reasoning about common graphs.

- (a) How many edges are in $K_{3,4}$? Is $K_{3,4}$ a regular graph?
12 / No, since some vertices are degree 3 and some degree 4
- (b) How many edges are in K_5 ? Is K_5 a regular graph?
10 / Yes, a 4-regular graph
- (c) What is the largest n such that $K_n = C_n$?
 $n=3$ The total degree of K_n is $n(n-1)$ while the total degree of C_n is $n(2)$ so $n(n-1)=2n$ only when $n=3$.
- (d) For what value of n is $Q_2 = C_n$?
 $n=3$
- (e) Is Q_n a regular graph for $n \geq 1$?
If so, what is the degree of the vertices in Q_n ?
next time...