Solutions HW 10

1) Done in HW 9, Ex 4.

2) Tu, 19.2:

$$F^*(u\,dv + v\,dv) = (x^2+y^2)\,d(x^2+y^2) + xy\,d(xy)$$
$$= (x^2+y^2)(2x\,dx+2y\,dy) + xy(y\,dx + x\,dy)$$
$$= (2x^3+3xy^2)\,dx + (2y^3+3x^2y)\,dy.$$

Tu, 19.11.

The form df is identically 0 on M (since $f\equiv 0$ on M). It can be expressed as

$$df = f_x\,dx + f_y\,dy.$$

Thus, an "orthogonal" form will do the job:

Let $\alpha = f_y\,dx - f_x\,dy$.

Then, α is smooth, so $\alpha|_M$ is also smooth, and if $0 \neq a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \in T_pM \subsetneq T_p\mathbb{R}^2$, we have $df\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) = af_x + bf_y = 0$, which implies $\alpha\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) = bf_x - af_y \neq 0$ since otherwise we would have $a=b=0$.

Thus, $\alpha = f_y\,dx - f_x\,dy$ is a nowhere vanishing form in M.

b) On M, $df = f_x\,dx + f_y\,dy + f_z\,dz \equiv 0$. If $f_x\neq 0$ and $f_y \neq 0$, this implies $-\frac{dx}{f_y} = \frac{dy}{f_x}$ on M.

Wedging with dz (on M), we get

$$\frac{dz\wedge dx}{f_y} = \frac{dy\wedge dz}{f_x}.$$

Similarly, if f_x and f_z or f_y and f_z are nonzero, we get a similar identity.
This shows that, whenever defined,

$$\frac{dx\wedge dy}{f_z} = \frac{dy\wedge dz}{f_x} = \frac{dz\wedge dx}{f_y}.$$

Thus, let

$$\alpha = \begin{cases} \dfrac{dx\wedge dy}{f_z} & \text{if } f_z \neq 0 \\ \dfrac{dy\wedge dz}{f_x} & \text{if } f_x \neq 0 \\ \dfrac{dz\wedge dx}{f_y} & \text{if } f_y \neq 0 \end{cases}$$

Then α is smooth and well defined.
To see that it is nonvanishing on M, assume WLOG, that $f_z \neq 0$ at some point. Then, if $\vec{u}, \vec{v} \in T_pM$, linearly independent, then

$$df(u) = f_x u_1 + f_y u_2 + f_z u_3 = 0 \Rightarrow$$

$$u_3 = -\frac{1}{f_z}(f_x u_1 + f_y u_2)$$

(similarly for $\vec{v}$).
This implies that, since the matrix $\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$ has rank 2, and since the last row is a linear combination of the first two, the first minor $u_1 v_2 - u_2 v_1 \neq 0$.
In particular,

$$\alpha(\vec{u}, \vec{v}) = \frac{1}{f_z}(u_1 v_2 - v_1 u_2) \neq 0.$$

This shows that α is nonvanishing.

(c) Use exactly the same procedure: show

$$\alpha = (-1)^i \frac{dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n}{f_{x^i}}$$ independent of i,

and exactly the same calculation shows that it is nonvanishing on M.

3) Tu, 20.7. Recall Cartan's magic formula:

$$\mathcal{L}_X\omega = i_X(d\omega) + d(i_X\omega).$$

Also, note that $i_{fX} = f i_X$. Thus,

$$\mathcal{L}_{fX}\omega = i_{fX}(d\omega) + d(i_{fX}\omega) = f\, i_X(d\omega) + d(f\, i_X\omega)$$
$$= f\, i_X(d\omega) + f\, d(i_X\omega) + df\wedge i_X\omega$$
$$= \underline{\mathcal{L}_X\omega + df\wedge i_X\omega.}$$

Tu, 20.10 Compute:

$$i_X w = w(X,\cdot) = x^2 dz + y^2 dy + xy\,dx - zy\,dy - zx\,dx$$
$$= x(y-z)dx + y(y-z)dy + x^2 dz.$$

(Notice that

$$x\,dy\wedge dz\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \cdot\right) =$$
$$x\,\underbrace{dy\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)}_{x} dz(\cdot) - x\,dy(\cdot)\,\overbrace{dz\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)}^{0}$$
$$= x^2 dz(\cdot).$$ The others are similar).

4) <u>Tu, 21.5</u>: $F(r,\theta) = (r\cos\theta, r\sin\theta)$.

We assume that the orientation on the domain is $[dr \wedge d\theta]$ and on the range is $dx \wedge dy$.

$$F^*(dx\wedge dy) = d(r\cos\theta)\wedge d(r\sin\theta)$$
$$= (\cos\theta\, dr - r\sin\theta\, d\theta)\wedge(\sin\theta\, dr + r\cos\theta\, d\theta)$$
$$= r\cos^2\theta\, dr\wedge d\theta + r\sin^2\theta\, dr\wedge d\theta$$
$$= r\, dr\wedge d\theta.$$

Since $r>0$, $[r\,dr\wedge d\theta] = [dr\wedge d\theta]$, so F is orientation preserving for the assumed orientations.

<u>21.9 Tu</u>: Recall from a previous homework that, on the tangent bundle, given charts $\tilde{\varphi}, \tilde{\psi}$, the determinant

$$\det\left(D\left(\tilde{\varphi}\circ\tilde{\psi}^{-1}\right)\right) = \left(\det\left(D(\varphi\circ\psi^{-1})\right)\right)^2 > 0.$$

This implies that $\{(\tilde{U}, \tilde{\varphi})\}$ (as constructed in class) is an oriented atlas, so TM is orientable.

5) There are several ways to do this. Here is one: Suppose $f^*\theta = 0$.

First notice that $d\theta = dy \wedge dz$, and $\theta \wedge d\theta = dx \wedge dy \wedge dz$. Also, $f^* d\theta = d f^*\theta = 0$.

If X, Y are linearly independent vectors on M and Z is a vector on $\mathbb{R}^3$,

$$\theta \wedge d\theta (f_* X, f_* Y, Z) = \theta(f_* X)\, d\theta(f_* Y, Z) - \theta(f_* Y)\, d\theta(f_* X, Z) + \theta(Z)\, d\theta(f_* X, f_* Y)$$

$$= (f^*\theta)(X)\, d\theta(f_* Y, Z) - (f^*\theta)(Y)\, d\theta(f_* X, Z) + \theta(Z)\,(f^* d\theta)(X, Y) = 0,$$

Since $\theta \wedge d\theta = dx \wedge dy \wedge dz$, which is nondegenerate, this implies that $f_* X, f_* Y, Z$ are linearly dependent for any choice of Z, which implies that $f_* X, f_* Y$ are linearly dependent. Therefore, f_* cannot have rank 2, so it is not an immersion.

6) a) M^m orientable if there is $\omega \in \Omega^m M$ w/ $\omega_p \neq 0$ $\forall p$.

b) If M and N orientable, then there are volume forms ω^M, ω^N in M and N. Then the form $(\omega^M \times \omega^N)(X^1, .., X^n, Y^1, .., Y^n) = \omega^M(X^1, .., X^n)\, \omega^N(Y^1, .., Y^n)$ is always nonzero in $M \times N$. Therefore $M \times N$ is orientable.

Conversely, if ω is a volume form in $M\times N$, fix $q\in N$, let $Y_1,\dots,Y_n\in T_qN$ be a frame, For any point $p\in M$, the point $(p,q)\in M\times N$, and $T_{(p,q)}(M\times N)=T_pM\times T_qN$. Consider the m form in M given by

$$\omega^M_p(X_1,\dots,X_m)=\omega_{(p,q)}(X_1,\dots,X_m,Y_1,\dots,Y_n).$$

Then ω^M is a nonvanishing top form on M, so M orientable

A similar argument shows N orientable.

7a) $[X,Y]=X(Y)-Y(X)=\left(x\frac{\partial}{\partial y}-x\frac{\partial}{\partial y}\right)-(x+y)(0)$

$=0.$

b) For X, $\left.\begin{array}{l} x'=x \\ y'=-x \end{array}\right\} \Rightarrow x=x_0e^t,\ y=-x_0e^t+(y_0+x_0)$
$=x_0(1-e^t)+y_0$

Thus, $\varphi_t(x,y)=(xe^t,-x(e^t-1)+y)$.

For Y, $\left.\begin{array}{l} x'=0 \\ y'=(x+y) \end{array}\right\} \Rightarrow x=x_0,\ y=(x_0+y_0)e^t-x_0$

Thus, $\psi_t(x,y)=(x,\ x(e^t-1)+ye^t)$

c) Let $F(s,t)=\varphi_s\circ\psi_t(1,0)=\varphi_s(1,e^t-1)$
$=(e^s,1-e^s+e^t-1)=(e^s,e^t-e^s)$.

It is easy to see that $F_{*(1,0)}$ is nonsingular, so F is a local diffeomorphism and F^{-1} will give a local chart around $(1,0)$.

Also,

$$\frac{\partial F}{\partial s} = (e^s, e^s) = (F_1(s,t), F_1(s,t))$$
$$= X_{F(s,t)}$$

and

$$\frac{\partial F}{\partial t} = (0, e^t) = (0, F_1(s,t) + F_2(s,t))$$
$$= Y_{F(s,t)}.$$

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8) We need to find $i_X(dx^1 \wedge \ldots \wedge dx^n)$, where $X_{(x^1, x^n)} = x^1 \frac{\partial}{\partial x^1} + \ldots + x^n \frac{\partial}{\partial x^n}$.

In general,

$$i_X(dx^1 \wedge \ldots \wedge dx^n) = dx^1(X)\, dx^2 \wedge \ldots \wedge dx^n - dx^2(X)\, dx^1 \wedge dx^3 \wedge \ldots \wedge dx^n + \ldots + (-1)^{n+1} dx^n(X)\, dx^1 \wedge \ldots \wedge dx^{n-1}.$$

Since $dx^i(X) = x^i$, we get

$$i_X \omega = x^1 dx^2 \wedge \ldots \wedge dx^n - x^2 dx^1 \wedge dx^3 \wedge \ldots \wedge dx^n + (-1)^{n+1} dx^1 \wedge \ldots \wedge dx^{n-1},$$

as claimed.

9) Using the orientation form ω^{S^n} of ex. 8, we get, since $\alpha(x^1,..,x^n)=(-x^1,..,-x^n)$,

$$\alpha^*\omega^{S^n} = \sum_{i=1}^{n+1} (-x^i)(-1)^n \, dx^1\wedge..\wedge \widehat{dx^i}\wedge..\wedge dx^{n+1}$$

$$= (-1)^{n+1}\sum_{i=1}^{n} x^i dx^1\wedge..\wedge\widehat{dx^i}\wedge..\wedge dx^{n+1}$$

$$= (-1)^{n+1}\omega^{S^n}.$$

Thus, α is orientation preserving if n odd, and reversing when n even.

(note that in ex. 8, the orientation form was for S^{n-1}, not for S^n.).

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10) Consider $\pi: S^n \to \mathbb{RP}^n$, n odd.

Note that $\pi\circ\alpha = \pi$, so

$$\pi^* = \alpha^*\circ\pi^*.$$

Now, $\pi^*: \Omega^n(\mathbb{RP}^n) \to \Omega^n(S^n)$.

Because π_* is onto, π^* is injective, and the image of π^* consists of forms $\theta\in\Omega^n(S^n)$ satisfying $\alpha^*\theta=\theta$.

Thus, ω^{S^n} is in the image of π^* and it has a single preimage $\tilde{\omega}^{\mathbb{RP}^n}$, which since $\pi^*\tilde{\omega}^{\mathbb{RP}^n} = \omega^{S^n}$, must be a volume form.

Thus, $\mathbb{RP}^n$ is orientable.