Homework 4. Solutions

2. $F(x,y) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =: A \begin{pmatrix} x \\ y \end{pmatrix}$

$(u,v) = F(x,y) = (x\cos\alpha - y\sin\alpha,\ x\sin\alpha + y\cos\alpha)$

$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. If $p=(x,y)$, and
$F_*(X_p) = a\frac{\partial}{\partial u}\Big|_{F(p)} + b\frac{\partial}{\partial v}\Big|_{F(p)}$, find a, b in terms of x, y, α.

Sol:

$$\frac{\partial}{\partial x}\Big|_p = \frac{\partial u}{\partial x}\Big|_p \frac{\partial}{\partial u}\Big|_p + \frac{\partial v}{\partial x}\Big|_p \cdot \frac{\partial}{\partial v}\Big|_p$$

$$= \cos\alpha \frac{\partial}{\partial u}\Big|_p + \sin\alpha \frac{\partial}{\partial v}\Big|_p \quad \text{(at any point } p\text{)}.$$

$$\frac{\partial}{\partial y}\Big|_p = -\sin\alpha \frac{\partial}{\partial u}\Big|_p + \cos\alpha \frac{\partial}{\partial v}\Big|_p$$

On the other hand,

$$F_{*p}\left(\frac{\partial}{\partial x}\right) = \frac{\partial F^1}{\partial x}\Big|_p \frac{\partial}{\partial x}\Big|_{F(p)} + \frac{\partial F^2}{\partial x}\Big|_p \frac{\partial}{\partial y}\Big|_{F(p)}$$

$$= \cos\alpha \frac{\partial}{\partial x}\Big|_{F(p)} + \sin\alpha \frac{\partial}{\partial y}\Big|_{F(p)}.$$ Similarly

$$F_{*p}\left(\frac{\partial}{\partial y}\right) = -\sin\alpha \frac{\partial}{\partial x}\Big|_{F(p)} + \cos\alpha \frac{\partial}{\partial y}\Big|_{F(p)}.$$

As a matrix, we can write

$$F_{*p}\left(a\frac{\partial}{\partial x}\Big|_p + b\frac{\partial}{\partial y}\Big|_p\right) = \left(\frac{\partial}{\partial x}\Big|_{F(p)} \ \frac{\partial}{\partial y}\Big|_{F(p)}\right) \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

and

$$\left(\frac{\partial}{\partial x}\Big|_{F(p)} \ \frac{\partial}{\partial y}\Big|_{F(p)}\right) = \left(\frac{\partial}{\partial u}\Big|_{F(p)} \ \frac{\partial}{\partial v}\Big|_{F(p)}\right) \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Thus, $F_*(X_p)$ is

$$\boxed{\left(\frac{\partial}{\partial u}\bigg|_{F(p)} \ \frac{\partial}{\partial v}\bigg|_{F(p)}\right)\begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix}}$$

NOTE: if we write $F(x,y) = (u,v)$, then, writing $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} =: A$, $\begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$, and

$$\begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A^{-1}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{pmatrix},$$

$$\left(F_*(X_p)\right)\Big|_{(u,v)} = \left(\frac{\partial}{\partial u}\bigg|_{(u,v)} \ \frac{\partial}{\partial v}\bigg|_{(u,v)}\right) \cdot \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$= \left(\frac{\partial}{\partial u}\bigg|_{(u,v)} \ \frac{\partial}{\partial v}\bigg|_{(u,v)}\right)\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} -v \\ u \end{pmatrix}.$$

[However, the exercise asks for an expression in terms of x, y, α].

Ex. 2 (cont).

Notice that there may be some confusion with this exercise. Our manifold M is $\mathbb{R}^2$.

We have two charts: $(\mathbb{R}^2, \underbrace{Id}_{\equiv \varphi} = (x,y))$ and $(\mathbb{R}^2, F(x,y) = (u,v))$ call it ψ.

Then we have a map $F: M \to M$, and a vector X. We want $F_*(X)$ on the basis $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$.

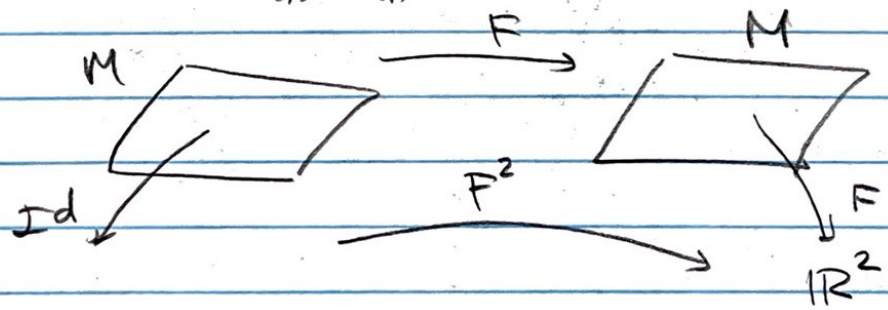
Since X is written on the basis $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$, what we want is the derivative

of

$$\overset{F}{\overset{|||}{\psi}} \circ F \circ \overset{Id}{\overset{|||}{\varphi^{-1}}}(r_1, r_2) = F \circ F(r_1, r_2)$$

since F is linear, $F_* = F$ and $(F \circ F)_* = F^2$, so what we have is

$$(F_* X)_{F(p)} = \left(\frac{\partial}{\partial u}\Big|_{F(p)}, \frac{\partial}{\partial v}\Big|_{F(p)}\right) F^2 \begin{pmatrix} -y \\ x \end{pmatrix}.$$



4) Write $\varphi_N = (x^1, x^2)$ and $\varphi_S = (y^1, y^2)$.

$$a_{*p}\left(a^1 \frac{\partial}{\partial x^1}\Big|_p + a^2 \frac{\partial}{\partial x^2}\Big|_p\right) = b^1 \frac{\partial}{\partial y^1}\Big|_{-p} + b^2 \frac{\partial}{\partial y^2}\Big|_{-p}.$$

We want to find b^1, b^2 as a function of a^1, a^2, p.

This function is the linear map

$$(\varphi_S \circ a \circ \varphi_N^{-1})_{*p}.$$

$$\varphi_N^{-1}(x^1, x^2) = \frac{1}{1+(x^1)^2+(x^2)^2}\left(2x^1, 2x^2, \underbrace{(x^1)^2+(x^2)^2}_{\text{call this } R^2} - 1\right)$$

$$\varphi_S(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

$$\Rightarrow \varphi_S \circ a \circ \varphi_N^{-1}(x^1, x^2) = \varphi_S \circ a\left(\frac{1}{1+R^2}(2x^1, 2x^2, R^2 - 1)\right)$$

$$= \varphi_S\left(\frac{-1}{1+R^2}(2x^1, 2x^2, R^2 - 1)\right)$$

$$= \left(\frac{\frac{-2x^1}{1+R^2}}{1+\frac{1-R^2}{1+R^2}}, \frac{\frac{-2x^2}{1+R^2}}{1+\frac{1-R^2}{1+R^2}}\right) = \frac{(-2x^1, -2y^1)}{2} = (-x^1, -x^2).$$

Thus, $(\varphi_S \circ a \circ \varphi_N^{-1})_{*p} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. In other words,

$$a_{*p}\left(a^1 \frac{\partial}{\partial x^1}\Big|_p + a^2 \frac{\partial}{\partial x^2}\Big|_p\right) = -a^1 \frac{\partial}{\partial y^1}\Big|_{(-p)} - a^2 \frac{\partial}{\partial y^2}\Big|_{(-p)}.$$

(5) $(S^2, \psi = (y^1, y^2))$. $(U_S, \varphi_S = (x^1, x^2))$

If $X_p = a^1 \frac{\partial}{\partial y^1}\Big|_p + a^2 \frac{\partial}{\partial y^2}\Big|_p = b^1 \frac{\partial}{\partial x^1}\Big|_p + b^2 \frac{\partial}{\partial x^2}\Big|_p$,

find (a^1, a^2) as a function of (b^1, b^2).
(or viceversa)

We need to find the derivative of

$$(\psi \circ \varphi_S^{-1})_{\varphi_S(p)}$$

$$\psi \circ \varphi_S^{-1}(x^1, x^2) = \psi\left(\frac{1}{1+R^2}(2x^1, 2x^2, -R^2+1)\right)$$

$$= \frac{1}{1+R^2}(2x^1, 2x^2) = \frac{1}{1+(x^1)^2+(x^2)^2}(2x^1, 2x^2)$$

Thus, $(\psi \circ \varphi_S^{-1})_{*(x^1,x^2)} = \frac{2}{(1+R^2)^2}\begin{pmatrix} 1-(x^1)^2+(x^2)^2 & -x^1x^2 \\ -x^1x^2 & 1+(x^1)^2-(x^2)^2 \end{pmatrix}$

At the point $\varphi_S(x, y, z)$,

$x^1 = \frac{x}{1+z}$, $x^2 = \frac{y}{1+z}$, substituting & simplifying,

we get

$$\begin{pmatrix} a^1 \\ a^2 \end{pmatrix} = \begin{pmatrix} 1-x^2+z & -\frac{xy}{2} \\ -\frac{xy}{2} & 1-y^2+z \end{pmatrix}\begin{pmatrix} b^1 \\ b^2 \end{pmatrix}.$$

6) a) Let $c(t)$ be a curve in V such that $c(0)=p$, $c'(0)=X\in V$. Then

$$L_{*_p}(X)=\frac{d}{dt}\Big|_0 L(c(t)) = \lim_{t\to 0}\frac{L(c(t))-L(c(0))}{t}=$$

(we are in a vector space)

$$= \lim_{t\to 0} L\left(\frac{c(t)-c(0)}{t}\right) = L\left(\lim_{t\to 0}\frac{c(t)-c(0)}{t}\right)$$

L linear — L continuous

$$= L(c'(0)) = L(X).$$

b) Let $c(t)=(c_1(t),\dots,c_k(t))$ be a curve in V^k with $c(0)=(p_1,\dots,p_k)$ and $c'(0)=(X^1,\dots,X^k)$.

$c(t)$ can be written as the composition of the following maps: let

$$r:\mathbb{R}\to\mathbb{R}^k,\quad r(t)=(\underbrace{t,t,\dots,t}_{k}),$$

let $\tilde{c}:\mathbb{R}^k\to V^k$, $\tilde{c}(t_1,\dots,t_k)=(c_1(t_1),\dots,c_k(t_k))$.

Then $c(t)=\tilde{c}\circ r(t)$.

$$L_{*_p}(X^1,\dots,X^k)=(L\circ c)'(0)=\sum_{i=1}^{k}\frac{\partial(L\circ\tilde{c})}{\partial t_i}\Big|_{t_i=0}\cdot\frac{dr_i}{dt}\Big|_{t=0}$$

$$\text{Now } \frac{\partial(L\circ\tilde{c})}{\partial t_i}\Big|_0 = \frac{d\big(L(c_1(0),c_2(0),\dots,c_i(t_i),\dots,c_k(0))\big)}{dt_i}\Big|_{t_i=0}$$

$$= L(p_1,p_2,\dots,X_i,\dots,p_k) \text{ (from part a).}$$

Thus, $L_{*_p}(X^1,\dots,X^k)=\sum_{i=1}^{k}L(p_1,p_2,\dots,X_i,\dots,p_k)$.

(7) Let $m : M_{n\times n}(\mathbb{R}) \times M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$,
$m(A, B) = AB$.
Then m is multilinear, and

$$\frac{d}{dt}\Big|_0 m(A(t), B(t)) = m_{*(A(0),B(0))}(X, Y) = XB(0) + A(0)Y,$$

from exercise 6b.

Alternative way (which will also work in Lie Groups)

Let $r(t) = (t, t)$, and $C(t_1, t_2) = A(t_1)B(t_2)$.

Then $\frac{d}{dt}\Big|_0 A(t)B(t) = \frac{d(c\circ r)}{dt}\Big|_0 \underset{\text{chain rule}}{=} XB(0) + A(0)Y$.

(8)

a) Let A be a curve in $Gl(n, \mathbb{R})$ s.t. $A(0) = e$, $A'(0) = X_e$. We need to find $i_{*e}(X) = \frac{d}{dt}\Big|_0 i(A(t))$.

Consider the curve $C(t) = A(t)\, i(A(t))$ $(= e)$.

Then $0 = \frac{dC}{dt}\Big|_0 = A'(0)\, \underbrace{i(e)}_{e} + e \cdot \frac{d\, i(A(t))}{dt}\Big|_0$

$= X + i_{*e}(X)$

$\Rightarrow \boxed{i_{*e}(X) = -X}$

b) Let $B(t)$ s.t. $B(0) = e$, $B'(0) = 0$, then
$c_{A*e}(X) = \frac{d}{dt}\Big|_0 AB(t)A^{-1} = AB'(0)A^{-1} = AXA^{-1}$.

c) $Ad_Y(A) = AYA^{-1}$. Let $A(t)$ be a curve with $A(0)=e$, $A'(0)=X$. By part a),

$$\frac{d}{dt}\Big|_0 (A(t))^{-1} = -X.$$

By exercise 7, $\frac{d}{dt}\Big|_0 \left((A(t)Y)(A(t))^{-1}\right)$

$$= \frac{d}{dt}\Big|_0 (A(t)Y)\cdot e + Y\frac{d}{dt}\Big|_0 (A(t))^{-1}$$

$$= XY - YX.$$

9) Let $X_p \in T_pM$ be a derivation. Then $X_p : C_p^\infty(M) \to \mathbb{R}$, and in particular, $X_p|_{F(p)} : F_p \subset C_p^\infty(M) \to \mathbb{R}$ is a linear map, and therefore $X_p \in F_p^*$.

On the other hand, because X_p is a derivation, if $\sum a_i [f_i][g_i] \in F_p$,

$$X_p\left(\sum a_i [f_i][g_i]\right) = \sum a_i \left(X([f_i])\overset{0}{\overbrace{g_i(p)}} + \overset{0}{\overbrace{f_i(p)}}X([g_i])\right)$$

$$= 0.$$

Thus, $X_p \in F_p^*$ and $X_p(F_p^2) = 0$, and therefore $X_p \in H_p^*$.

b) Let $h \in H_p^*$. Let $D_h : C_p^\infty(M) \to \mathbb{R}$, $D_h([f]) = h([f - f(p)])$.

Then

$$D_h([f][g]) = h([fg - f(p)g(p)])$$ (note – I'll drop the "[ ]")

$$= h\Big(\underbrace{(f-f(p))(g-g(p))}_{\in F_p^2 \subset F_p} + \underbrace{fg(p) + f(p)g - 2f(p)g(p)}_{\in F_p}\Big)$$

$$= h((f-f(p))g(p)) + h(f(p)(g-g(p))) = \longrightarrow \text{over}$$

$h(F_p^2) = 0$

$g(p)\,h(f-f(p)) + f(p)\,h(g-g(p))$

$= D_h(f)\,g(p) + f(p)\,D_h(g)$, as desired.

c) If $X_p \in T_pM$, $\forall f \in C_p^\infty(M)$, $D_{X_p}(f) = X_p(f-f(p)) \underset{X_p(f(p))=0}{=} X_p(f)$,
so $D_{X_p} = X_p$.

If $h \in H_p^*$, and $f \in F_p/F_p^2$, then

$$D_h(f) = h(f - \underset{0}{f(p)}) = h(f).$$

(this shows the two maps above are inverses of each other).