Hw 2. Solution

(1) $\alpha = xy\,dx + xzy\,dy - 3w^2\,dz + 5y\,dw$

$\beta = y^2z\,dx\wedge dy + zw\,dx\wedge dz - 5x\,dz\wedge dw - xyzw\,dy\wedge dw.$

a) $d\alpha = -x\,dx\wedge dy + zy\,dx\wedge dy - xy\,dy\wedge dz + 6w\,dz\wedge dw + 5y\,dy\wedge dw.$

b) $d\beta = y^2\,dx\wedge dy\wedge dz + z\,dx\wedge dz\wedge dw - 5\,dx\wedge dz\wedge dw$

$\quad - yzw\,dx\wedge dy\wedge dw + xyw\,dy\wedge dz\wedge dw.$

c) $\alpha\wedge\beta = -5x^2y\,dx\wedge dz\wedge dw + x^2y^2zw\,dx\wedge dy\wedge dw$

$\quad - xz^2yw\,dx\wedge dy\wedge dz - 5x^2zy\,dy\wedge dz\wedge dw$

$\quad - 3y^2zw^2\,dx\wedge dy\wedge dw - 3xyzw^3\,dy\wedge dz\wedge dw$

$\quad + 5y^3z\,dx\wedge dy\wedge dw + 5yzw\,dx\wedge dz\wedge dw$

d) $\alpha\wedge d\beta = x^2y^2w\,dx\wedge dy\wedge dz\wedge dw - xyz(z-5)\,dx\wedge dy\wedge dz\wedge dw$

$\quad + 3yzw^3\,dx\wedge dy\wedge dz\wedge dw - 5y^3\,dx\wedge dy\wedge dz\wedge dw.$

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(2) Write α as $\alpha = a\,dx\wedge dy + b\,dy\wedge dz + c\,dx\wedge dz.$

Then $a = \alpha\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$, $b = \alpha\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, $c = \alpha\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)$.

$$a = \alpha_p\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = p\cdot\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix} = z.$$

$$c = \alpha_p\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = p\cdot\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = -y$$

$$b = \alpha_p\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = p\cdot\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = x.$$

Therefore, $\alpha = z\,dx\wedge dy + x\,dy\wedge dz - y\,dx\wedge dz$.

$$d\alpha = dx\wedge dy\wedge dz + dx\wedge dy\wedge dz + dx\wedge dy\wedge dz$$
$$= 3\,dx\wedge dy\wedge dz.$$

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3 (a) Suppose $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of ΛV.

Then

- All elements of $\Lambda^1 V$ are decomposable by definition.
- " " " $\Lambda^3 V$ " " "
- Elements of $\Lambda^2 V$ have the form

$$\alpha = a\alpha_1\wedge\alpha_2 + b\alpha_1\wedge\alpha_3 + c\alpha_2\wedge\alpha_3.$$

a) If $b=0$, then $\alpha = (a\alpha_1 - c\alpha_3)\wedge\alpha_2$, which is decomposable.

b) If $b\neq 0$, then $\alpha = (b\alpha_1 + c\alpha_2)\wedge(\alpha_3 + \frac{a}{b}\alpha_2)$, which is also decomposable.

b) Note that if α is homogeneous, $\alpha\wedge\alpha = 0$, because

$$\alpha\wedge\alpha = \underbrace{\alpha_1\wedge\ldots\wedge\alpha_k\wedge\alpha_1}_{\text{repeated}}\wedge\ldots\wedge\alpha_k = 0.$$

If $\dim(V) = 0$, and $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a basis, then $\alpha = \alpha_1\wedge\alpha_2 + \alpha_3\wedge\alpha_4$ cannot be decomposable, because $\alpha\wedge\alpha = 2\,\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4$.

(4) Given the k linearly independent forms $\{\omega^1, \dots, \omega^k\}$ extend this set to a basis $\{\omega^1, \dots, \omega^n\}$.

Then $\theta^i = \sum_{j=1}^{n} A^i_j \omega^j$.

Since $\sum_{i=1}^{k} \theta^i \wedge \omega^i = 0$, we have

$$\sum_{i=1}^{k} \sum_{j=1}^{n} A^i_j \, \omega^j \wedge \omega^i = 0. \quad \text{We rewrite the sum:}$$

$$\sum_{i=1}^{k} \sum_{j=1}^{n} A^i_j \, \omega^j \wedge \omega^i = \sum_{i=1}^{k} \sum_{j=1}^{k} A^i_j \, \omega^j \wedge \omega^i + \sum_{i=1}^{k} \sum_{j=k+1}^{n} A^i_j \, \omega^j \wedge \omega^i$$

$$= \sum_{1 \le i < j \le k} \left(A^j_i - A^i_j\right) \omega^i \wedge \omega^j - \sum_{i=1}^{k} \sum_{j=k+1}^{n} A^i_j \, \omega^i \wedge \omega^j.$$

Since $\{\omega^i \wedge \omega^j\}_{1 \le i < j \le k} \cup \{\omega^i \wedge \omega^j\}_{\substack{1 \le i \le k \\ k+1 \le j \le n}}$ is a linearly independent set (is a subset of a basis), we must have $A^j_i = A^i_j$ for $1 \le i < j \le k$, and $A^i_j = 0$ for $1 \le i \le k$, $k+1 \le j \le n$. Therefore

$$\theta = \sum_{j=1}^{n} A^i_j \omega^j = \sum_{j=1}^{k} A^i_j \omega^j + \sum_{j=k+1}^{n} \cancelto{0}{A^i_j} \omega^j$$

$$= \sum_{j=1}^{k} A^i_j \omega^j, \text{ with } A^i_j = A^j_i,$$

as claimed.

(5) Prove $\Lambda^p V \xrightarrow{\wedge\xi} \Lambda^{p+1} V \xrightarrow{\wedge\xi} \Lambda^{p+2} V$

Let us denote by m_p the map $m_p : \Lambda^p V \to \Lambda^{p+1} V$, $m_p(\alpha) = \alpha \wedge \xi$. We need to show

1) $m_p(\Lambda^p V) \subset \ker(m_{p+1})$

2) $\ker(m_{p+1}) \subset m_p(\Lambda^p V)$.

1) If $\beta \in m_p(\Lambda^p V)$, $\beta = m_p(\alpha) = \alpha \wedge \xi$. But then $\beta \in \ker(m_{p+1})$ because $m_{p+1}(\beta) = \alpha \wedge \xi \wedge \xi = 0$.

2) By exercise 7 of Homework 1, if $\xi \wedge \beta = 0$, then $\beta = \xi \wedge \tau$ for some $\tau \in \Lambda^p V$, so that $\beta \in m_p(\Lambda^p V)$.

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(6) Let $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \in \mathfrak{X}(\mathbb{R}^3)$.

$\operatorname{curl} X = (c_y - b_z)\frac{\partial}{\partial x} + (a_z - c_x)\frac{\partial}{\partial y} + (b_x - a_y)\frac{\partial}{\partial z}$.

On the other hand,

$$\# \circ * \circ d \circ \flat\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\right) = \# \circ * \circ d\,(a\,dx + b\,dy + c\,dz)$$

$$= \# \circ * \big((b_x - a_y)\,dx\wedge dy + (a_z - c_x)\,dz\wedge dx + (c_y - b_z)\,dy\wedge dz\big)$$

$$= \# \big((b_x - a_y)\,dz + (a_z - c_x)\,dy + (c_y - b_z)\,dx\big)$$

$$= (b_x - a_y)\frac{\partial}{\partial z} + (a_z - c_x)\frac{\partial}{\partial y} + (c_y - b_z)\frac{\partial}{\partial x},$$

as desired.

(5 cont)

$$\operatorname{div} X = a_x + b_y + c_z.$$

$$\ast \circ d \circ \ast \circ \flat(X) = \ast \circ d \circ \ast\,(a\,dx + b\,dy + c\,dz)$$

$$= \ast \circ d\,(a\,dy \wedge dz + b\,dz \wedge dx + c\,dx \wedge dy)$$

$$= \ast((a_x + b_y + c_z)\,dx \wedge dy \wedge dz$$

$$= a_x + b_y + c_z, \text{ as desired}$$

$$\Delta f = f_{xx} + f_{yy} + f_{zz}$$

$$d \circ \ast \circ d \circ \ast(f) = d \circ \ast \circ d\,(f\,dx \wedge dy \wedge dz) = 0$$

$$\ast \circ d \circ \ast \circ d\,(f) = \ast \circ d \circ \ast\,(f_x\,dx + f_y\,dy + f_z\,dz)$$

$$= \ast \circ d\,(f_x\,dy \wedge dz + f_y\,dz \wedge dx + f_z\,dx \wedge dy)$$

$$= \ast((f_{xx} + f_{yy} + f_{zz})\,dx \wedge dy \wedge dz$$

$$= f_{xx} + f_{yy} + f_{zz}, \text{ as desired}$$