Differential Geometry. Homework 8. Due April 28th. Professor: Luis Fernández

NOTE: if you need, please ask for hints.

1. Finish all the homework assignments up to this point so that you can hand them in after the break.

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2. Use your favorite computer algebra software (Maple, Mathematica) to write a program that calculates the Gaussian curvature of a surface in $\mathbb{R}^3$ given the parametrization. That is, if M is the subset of R^3 parametrized by $\varphi(u,v)$ (i.e. $\varphi : U \subset \mathbb{R}^2 \to M \subset \mathbb{R}^3$ is a diffeomorphism over its image), then define the metric, as usual, by

$$g_{11} = \left\langle \frac{\partial \varphi}{\partial u}, \frac{\partial \varphi}{\partial u} \right\rangle_{\mathbb{R}^3} \qquad g_{12} = g_{21} = \left\langle \frac{\partial \varphi}{\partial u}, \frac{\partial \varphi}{\partial v} \right\rangle_{\mathbb{R}^3} \qquad g_{22} = \left\langle \frac{\partial \varphi}{\partial v}, \frac{\partial \varphi}{\partial v} \right\rangle_{\mathbb{R}^3},$$

and from here find the Christoffel symbols, and from there find the curvature.

Try to find some geodesics too, at least numerically.

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3. (This is in Lee's Riemannian Manifolds, ex 7-2) Let ∇ be the Riemannian connection on a Riemannian manifold (M,g), and let ω_i^j be its connection 1-forms with respect to a local frame $\{E_i\}$ (i.e. satisfying $\nabla_X E_i = \sum_r \omega_i^r(X) E_r$ for every vector X). Define a matrix of 2-forms Ω_i^j, called the curvature 2-forms, by

$$R(X,Y)E_i = \sum_j \Omega_i^j(X,Y)E_j,$$

or, in terms of the local moving frame,

$$\Omega_i^j = \frac{1}{2}\sum_{r,s} R_{rsi}^j\, \theta^r \wedge \theta^s,$$

where $\{\theta^i\}_{i=1}^n$ is the dual frame of $\{E_i\}_{i=1}^n$ (i.e. $\theta^i(E_j) = \delta_j^i$). Show that they satisfy Cartan's second structural equation:

$$d\omega_i^j = -\sum_r \omega_r^j \wedge \omega_i^r + \Omega_i^j.$$

Further, if the frame is orthonormal, then $\omega_i^j = -\omega_j^i$.

(Recall that Cartan's first structure equation was proved in Hw 6; in this case the torsion $\tau = 0$ because we are using the Riemannian connection. These two equations together:

$$d\theta^i = -\sum_r \omega_r^i \wedge \theta^r, \qquad d\omega_i^j = -\sum_r \omega_r^j \wedge \omega_i^r + \Omega_i^j, \quad \text{with } \omega_i^j = -\omega_j^i$$

are extremely useful, for example, to find the curvature of a manifold).

[HINT: Remember the useful formula: if α is a 1-form and $X, Y \in \mathfrak{X}(M)$, then

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]).$$

Use this formula to find $d\omega_j^i(E_k, E_\ell)$ and note that $\omega_j^i(E_\ell) = \langle \nabla_{E_\ell} E_j, E_i \rangle$.]

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4. Use the structure equations to show that the sectional curvature of the hyperbolic space $\mathbb{H}^n$ is -1, as follows:

Recall that $\mathbb{H}^n = \{(x^1, \ldots, x^n) : x^n > 0\}$, with the metric given by

$$g = \frac{1}{(x^n)^2} \sum_i dx^i \otimes dx^i.$$

(That is, $\langle X, Y \rangle^{\mathbb{H}^n} = \frac{1}{(x^n)^2}\, X \cdot Y$, where "$\cdot$" denotes the dot product in $\mathbb{R}^n$.

If we let $e_i = x^n \dfrac{\partial}{\partial x^i}$, then $\{e_i\}_{i=1}^n$ is an orthonormal frame in $\mathbb{H}^n$. The dual coframe $\{\theta^i\}_{i=1}^n$ is given by $\theta^i = x^n dx^i$.

a) Find $d\theta^i$, write it in terms of the coframe $\{\theta^i\}_{i=1}^n$, and use the first structure equation to show that $\omega_i^j = 0$ if $i, j \neq n$ and $\omega_i^n = -\omega_n^i = \theta^i$.

b) Then compute $d\omega_j^i$ and use the second structure equation to show that $\Omega_j^i = -\theta^i \wedge \theta^j$.

c) Finally, Use the definition of the curvature form in terms of R, and the definition of the sectional curvature K to show that $K = -1$.

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5. (The pseudosphere) Consider the surface in $\mathbb{R}^3$ parametrized by

$$\varphi(u, v) = (\text{sech}u \cos v, \text{sech}u \sin v, u - \tanh u).$$

Prove that it has Gaussian curvature -1 (of course, with the metric induced from $\mathbb{R}^3$).

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6. (DoCarmo, Chapter 4, Exercise 1.) Let G be a Lie group and $\mathfrak{g}$ its Lie algebra, and let g be a bi-invariant metric on G. In the previous homework you showed that if ∇ is the Levi-Civita connection for this metric, then $\nabla_X Y = \dfrac{1}{2}[X, Y]$ whenever X and Y are left-invariant vector fields on G.

Suppose that X, Y, Z are left-invariant vector fields on G.

a) Prove that $R(X, Y)Z = \dfrac{1}{4}[[X, Y], Z]$

b) Prove that if X and Y are orthonormal, then the sectional curvature $K(\sigma)$ of G, where σ is the plane generated by X and Y is given by

$$K(\sigma) = \frac{1}{4}\|[X, Y]\|^2.$$

Therefore the sectional curvature $K(\sigma)$ of a Lie group with bi-invariant metric is non-negative, and it is zero fid and only if σ is generated by vectors X, Y that commute, i.e. such that $[X, Y] = 0$.

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7. (DoCarmo, Ex. 6 of Chapter 5): Let M be a Riemannian manifold of dimension two (in this case we say that M is a surface). Let $B_\delta(p)$ be a normal ball around the point $p \in M$ and consider the parametrized surface

$$f(\rho, \theta) = \exp_p \rho v(\theta), \quad 0 < \rho < \delta, \quad -\pi < \theta < \pi,$$

where $v(\theta)$ is a circle of radius δ in T_pM parametrized by the central angle θ.

a) Show that (p, θ) are coordinates in an open set $U \subset M$ formed by the open ball $B_\delta(p)$ minus the ray $\exp_p(-\rho v(0))$, $0 < \rho < \delta$. Such coordinates are called polar coordinates at p.

b) Show that the coefficients g_{ij} of the Riemannian metric in these coordinates are:

$$g_{12} = 0 \quad g_{11} = \left|\frac{\partial f}{\partial \rho}\right|^2 = |v(\theta)|^2 = 1, \quad g_{22} = \left|\frac{\partial f}{\partial \theta}\right|^2$$

c) Show that, along the geodesic $f(\rho, 0)$, we have

$$(\sqrt{g_{22}})_{\rho\rho} = -K(p)\rho + R(\rho), \quad \text{where} \quad \lim_{\rho \to 0} \frac{R(\rho)}{\rho} = 0,$$

and $K(\rho)$ is the sectional curvature of M at p.

d) Prove that

$$\lim_{\rho \to 0} \frac{\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}})} = -K(p).$$

This last expression is the value of the Gaussian curvature of M at p given in polar coordinates (Cf., for example, M. do Carmo [dC 2] p. 288). This fact from the theory of surfaces, and (d) shows that, in dimension two, the sectional curvature coincides with the Gaussian curvature. In the next chapter, we shall give a more direct proof of this fact.

8. (DoCarmo, Ex. 7 of Chapter 5): Let M be a Riemannian manifold of dimension two. Let $p \in M$ and let $V \subset T_pM$ be a neighborhood of the origin where $\exp_p$ is a diffeomorphism. Let $S_r(0) \subset V$ be a circle of radius r centered at the origin, and let L_r be the length of the curve $\exp_p(S_r)$ in M. Prove that the sectional curvature at $p \in M$ is given by

$$K(p) = \lim_{r \to 0} \frac{3}{\pi} \frac{2\pi r - L_r}{r^3}.$$

[Hint: Use Exercise 6.]