Differential Geometry. Homework 7. Due April 7th. Professor: Luis Fernández

NOTE: if you need, please ask for hints.

1. Consider $\mathbb{H}^{\not\vDash}$ with the half plane model, i.e. $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, with metric $g = \dfrac{1}{y^2}(dx \otimes dx + dy \otimes dy)$.

a) Find the Christoffel symbols for the Levi-Civita connection.

b) Find the equation of the geodesics.

c) Find explicitly a parametrization of the geodesic emanating from the point $(0, 1)$ with initial velocity (u, v).

d) Taking $(u, v) = (\cos\theta, \sin\theta)$, find a parametrization of the geodesic circle of radius r with center $(0, 1)$.

e) Find the circumference of this circle.

f) Find the area of this circle.

NOTE: You are welcome to use Mathematica or any algebra program to do this exercise. You can then print the outputs and attach it to the homework.

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2. Consider S^2 with the metric inherited from $\mathbb{R}^3$.

a) Find explicitly a parametrization of the geodesic emanating from the north pole with initial velocity $(u, v, 0)$ (if you prefer, you can use any other point instead of the north pole).

b) Taking $(u, v) = (\cos\theta, \sin\theta)$, find a parametrization of the geodesic circle of radius r with center at the north pole.

c) Find the circumference of this circle.

d) Find the area of this circle.

NOTE: You are welcome to use Mathematica or any algebra program to do this exercise. You can then print the outputs and attach it to the homework.

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3. (Lee 2, p. 89, 5-11.) Let G be a Lie group and $\mathfrak{g}$ its Lie algebra, and let g be a bi-invariant metric on G.

a) For any $X, Y, Z \in \mathfrak{g}$, show that

$$\langle [X, Y], Z\rangle = -\langle Y, [X, Z]\rangle.$$

[Hint: Let $\gamma(t) = \mathrm{Exp}(tX)$ (here Exp denotes the Lie group exponential map, not the Riemannian one), and compute the t- derivative of $\langle Ad_{\gamma(t)}Y, Ad_{\gamma(t)}Z\rangle$ at $t = 0$, using the facts that $Ad_{\gamma(t)} = (R_{\gamma(t)})_*(L_{\gamma(t)})_*$ and that $R_{\gamma(t)}$ is the flow of $-X$.]

b) Show that

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

whenever X and Y are left-invariant vector fields on G.

[Hint: recall that we had the relation $2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle + \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle - \langle [Y, Z], X\rangle$. If X, Y, Z are left invariant, and $\langle\ ,\ \rangle$ is bi-invariant, then this expression simplifies considerably, and you can easily prove that $\nabla_X X = 0$.]

c) Show that the geodesics of g starting at the identity are exactly the one-parameter subgroups, so the Lie group exponential map coincides with the Riemannian exponential map at the identity.

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4. (From Lee, Riemannian Manifolds, Ex. 5-5) Let (M, g) be a Riemannian manifold. If f is a smooth function on M such that $|\mathrm{grad} f| \equiv 1$, show that the integral curves of $\mathrm{grad} f$ are geodesics.

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5. (DoCarmo, Chapter 3, Ex. 11) In a previous homework we defined the divergence of a vector field $X \in \mathfrak{X}(M)$ by

$$d(i_X\omega) = (\mathrm{div}\, X)\, \omega,$$

where ω is the Riemannian volume element, that is $\omega = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n$.

Prove that $\operatorname{div} X$ equals the trace of the operator

$$Y \to \nabla_Y X.$$

(See the hint in DoCarmo, Chapter 3, Ex. 11.)

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6. Lee 5-1.

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7. Lee 5-2 (similar to DoCarmo p. 78, Ex. 1).

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8. Lee 5-4.
