Differential Geometry. Homework 6. Due March 31st. Professor: Luis Fernández

NOTE: if you need, please ask for hints.

1. Let Γ^k_{ij} be the Christoffel symbols for the chart $(U, \varphi = (x^1, \ldots, x^n))$ and Γ'^k_{ij} be the Christoffel symbols for the chart $(V, \psi = (y^1, \ldots, y^n))$ (i.e. $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}$, similarly with y). Prove that in $U \cap V$ the change of basis is given by

$$\Gamma'^r_{ij} \frac{\partial x^k}{\partial y^r} = \Gamma^k_{st} \frac{\partial x^s}{\partial y^i} \frac{\partial x^t}{\partial y^j} + \frac{\partial^2 x^k}{\partial y^i \partial y^j}.$$

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2. Consider a manifold M with charts $(U, \varphi = (x^1, \ldots, x^n))$ and $(V, \psi = (y^1, \ldots, y^n))$. Let $p_M : TM \to M$ be the usual projection taking every vector in T_pM to p and let $\widetilde{U} = p_M^{-1}(U)$, $\widetilde{V} = p_M^{-1}(V)$. Consider then the manifold TM with charts $(\widetilde{U}, \widetilde{\varphi})$ and $(\widetilde{V}, \widetilde{\psi})$ given by

$$\widetilde{\varphi}(p, \vec{v}_p) = (\varphi(p), d\varphi_p(\vec{v}_p)) \quad \text{and} \quad \widetilde{\psi}(p, \vec{v}_p) = (\psi(p), d\psi_p(\vec{v}_p)),$$

i.e. if $\vec{v}_p = a^i \frac{\partial}{\partial x^i}_{|p}$ then

$$\widetilde{\varphi}(p, \vec{v}_p) = (x^1(p), \ldots, x^n(p), a^1, \ldots, a^n),$$

which we will write as (x^i, a^i). Similarly, if $\vec{v}_p = b^i \frac{\partial}{\partial y^i}_{|p}$ then

$$\widetilde{\psi}(p, \vec{v}_p) = (y^i, b^i).$$

a) Show that if $\vec{v}_p = a^i \frac{\partial}{\partial x^i}_{|p} = b^i \frac{\partial}{\partial y^i}_{|p}$, then

$$b^i = a^j \frac{\partial y^i}{\partial x^j}.$$

Since (x^i, a^i) are coordinates in TM, $\{\frac{\partial}{\partial x^i}_{|\vec{v}_p}, \frac{\partial}{\partial a^i}_{|\vec{v}_p}\}_{i=1}^n$ is a basis of $T_{\vec{v}_p}TM$ (where as usual if $f \in C^\infty(TM)$ then $\frac{\partial}{\partial x^i}_{|\vec{v}_p}(f) = \frac{\partial (f \circ \widetilde{\varphi}^{-1})}{\partial x^i}$ and $\frac{\partial}{\partial a^i}_{|\vec{v}_p}(f) = \frac{\partial (f \circ \widetilde{\varphi}^{-1})}{\partial a^i}$). Similarly, $\{\frac{\partial}{\partial y^i}_{|\vec{v}_p}, \frac{\partial}{\partial b^i}_{|\vec{v}_p}\}_{i=1}^n$ are a basis of $T_{\vec{v}_p}TM$.

b) Prove that

$$\frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} + a^j \frac{\partial^2 y^r}{\partial x^i \partial x^j} \frac{\partial}{\partial b^r} \quad \text{and} \quad \frac{\partial}{\partial a^i} = \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial b^j}.$$

c) Using exercise 1 conclude that the geodesic vector field defined locally by

$$G = a^k \frac{\partial}{\partial x^k} - \Gamma^k_{ij} a^i a^j \frac{\partial}{\partial a^k}$$

where Γ^k_{ij} are the Christoffel symbols for the x^i, is independent of the choice of coordinates, i.e. in the basis (y^i, b^i),

$$G = b^k \frac{\partial}{\partial y^k} - \Gamma'^k_{ij} b^i b^j \frac{\partial}{\partial b^k}$$

where Γ'^k_{ij} are the Christoffel symbols for the y^i.

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3. Lee, Riemannian manifolds, exercise 4.4: Let ∇ be a linear connection. If ω is a 1-form and X a vector field, show that the coordinate expression for $\nabla_X \omega$ is

$$\nabla_X \omega = \left(X^i \frac{\partial \omega_k}{\partial x^i} - X^i \omega_j \Gamma^j_{ik} \right) dx^k,$$

where $\omega = \sum \omega_k dx^k$ and Γ^k_{ij} are the Christoffel symbols of ∇ for the given coordinate system $(x^1, \ldots, x^n)$.

4. (This is from Lee's 'Riemannian Manifolds', exercise 3-3) Let (M, g) be an oriented Riemannian manifold with volume element ω. The divergence operator div : $\mathfrak{X}(M) \to C^\infty(M)$ is defined by

$$d(i_X\omega) = (\operatorname{div} X)\,\omega,$$

where i_X denotes interior multiplication by X.

a) Suppose M is a compact, oriented Riemannian manifold with boundary. Prove the following divergence theorem for $X \in \mathfrak{X}(M)$:

$$\int_M (\operatorname{div} X)\,\omega = \int_{\partial M} \langle X, N\,\rangle\, i_N\omega,$$

where N is the outward unit normal to ∂M and $i_N\omega$ is interior product.

b) Show that the divergence operator satisfies the following product rule for a smooth function $u \in C^\infty(M)$:

$$\operatorname{div}(uX) = u \operatorname{div} X + \langle \operatorname{grad} u, X\rangle.$$

Deduce the following 'integration by parts' formula:

$$\int_M \langle \operatorname{grad} u, X\rangle\,\omega = -\int_M u \operatorname{div} X\;\omega + \int_{\partial M} u\,\langle X, N\rangle\, i_N\omega.$$

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5. Lee Riemannian Manifolds, p. 63, ex. 4-2: Let be a linear connection on M, and define a map $\tau : T(M)T(M)T(M)$ by

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

a) Show that τ is a $\binom{2}{1}$-tensor field, called the torsion tensor of ∇.

b) We say that ∇ is symmetric if its torsion vanishes identically. Show that ∇ is symmetric if and only if its Christoffel symbols with respect to any *coordinate* frame are symmetric: $\Gamma^k_{ij} = \Gamma^k_{ji}$ [Warning: They might not be symmetric with respect to other (non-coordinate) frames.]

c) Show that ∇ is symmetric if and only if the covariant Hessian ∇^2 of any smooth function $u \in C^\infty(M)$ is a symmetric 2-tensor field (see the definition of the covariant Hessian on page 54 of Lee).

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6. Let ∇ be a connection on M, let $\{E_i\}$ be a local frame on some open subset $U \subset M$, and let $\{\theta^j\}$ be the dual coframe.

a) Show that there is a uniquely determined matrix of 1-forms ω^i_j on U, called the connection 1-forms for this frame such that, for all $X \in TM$,

$$\nabla_X E_i = \omega^j_i(X)\, E_j,$$

b) Prove Cartan's first structure equation:

$$d\theta^j = -\sum_{i=1}^n \omega^j_i \wedge \theta^i + \tau^j,$$

where τ^i are the torsion 2-forms, defined in terms of the torsion tensor by

$$\tau(X, Y) = \sum_{j=1}^n \tau^j(X, Y)\, E_j.$$

7. (This is from Lee's 'Riemannian Manifolds', exercise 3-4). Let (M, g) be a compact, connected, oriented Riemannian manifold with boundary and with volume form ω. For $u \in C^\infty(M)$, the Laplacian of u, denoted Δu, is defined to be the function

$$\Delta u = \operatorname{div}(\operatorname{grad} u).$$

A function $u \in C^\infty(M)$ is said to be harmonic if $\Delta u = 0$.

a) Prove Green's identities:

$$\int_M u\, \Delta v\, \omega + \int_M \langle \operatorname{grad} u, \operatorname{grad} v \rangle\, \omega = \int_{\partial M} u\, N(v)\, i_N \omega.$$

$$\int_M (u\, \Delta v - v\, \Delta u)\, \omega = \int_{\partial M} (u\, N(v) - v\, N(u))\, i_N \omega.$$

b) If $\partial M \neq \emptyset$, and u, v are harmonic functions on M whose restrictions to ∂M agree, show that $u \equiv v$.

c) If $\partial M = \emptyset$, show that the only harmonic functions on M are the constants.

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8. DoCarmo, Chapter 1, Ex. 4.

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9. DoCarmo, Chapter 1, Ex. 7.