Differential Geometry. Homework 5. Due March 17th. Professor: Luis Fernández

1. Prove that the set of local sections of a vector bundle E over a manifold M is a sheaf:

Given X a topological space, a *sheaf* $\mathcal{F}$ on X associates to each open set $U \subset X$ a group $\mathcal{F}(U)$, called the, sections of $\mathcal{F}$ over U, and to each pair $U \subset V$ of open sets a map $r_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$, called the restriction map, satisfying,

1. For any triple $U \subset V \subset W$ of open sets,
$$r_{W,U} = r_{V,U} \circ r_{W,V}.$$
By virtue of this relation, we may write $\sigma|_U$ for $r_{V,U}(\sigma)$ without loss of information.
2. For any pair of open sets $U, V \subset X$ and sections $\sigma \in \mathcal{F}(U)$, $\tau \in \mathcal{F}(V)$ such that
$$\sigma|_{U\cap V} = \tau|_{U\cap V}$$
there exists a section $\rho \in \mathcal{F}(U \cup V)$ with
$$\rho|_U = \sigma \qquad \rho|_V = \tau.$$
3. If $\sigma \in \mathcal{F}(U \cup V)$ and
$$\sigma|_U = \sigma|_V = 0,$$
then $\sigma = 0$.

(In this case, to and open $U \subset M$ we associate the vector space $\Gamma(U, E)$ of local smooth sections of E over U and the restriction map is just restriction in the usual sense.)

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2. Lee, exercise 10-7: Compute the transition function for TS^2 associated with the two local trivializations determined by stereographic coordinates.

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3. Lee 10-2: Let E be a vector bundle over a topological space M. Show that the projection map $\pi : E \to M$ is a homotopy equivalence. That is, there exists a map $s : M \to E$ such that $\pi \circ s$ is homotopic to the identity in M and $s \circ \pi$ is homotopic to the identity in E. (Recall that two maps $f, g : M \to M$ are homotopic if there is a continuous map $F : [0, 1] \times M \to M$ such that $F(0, p) = f(p)$ and $F(1, p) = g(p)$.)

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4. Lee 10-12: Let $\pi : E \to M$ and $\pi' : E' \to M$ be two smooth rank k vector bundles over a smooth manifold M. Suppose that $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M such that both E and E' admit smooth local trivializations over each U_α. Let $\{\sigma_{\alpha\beta}\}$ an $\{\sigma'_{\alpha\beta}\}$ denote the transition functions determined by the given local trivializations of E and E', respectively. Show that E and E' are smoothly isomorphic over M if and only if for each $\alpha \in A$ there exists a smooth map $f_\alpha : U_\alpha \to Gl(k, \mathbb{R})$ such that
$$\sigma'_{\alpha\beta}(p) = f_\alpha(p)\, \sigma_{\alpha\beta}(p)\, (f_\beta(p))^{-1}.$$
In particular, taking E as the trivial bundle $M \times \mathbb{R}^k$ (so the transition functions are $\sigma_{\alpha\beta}(p) = I_{\mathbb{R}^n}$) , this shows that a bundle E' is trivial if for each $\alpha \in A$ there exists a smooth map $f_\alpha : U_\alpha \to Gl(k, \mathbb{R})$ such that $\sigma_{\alpha\beta}(p) = f_\alpha(p)\, (f_\beta(p))^{-1}$.

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5. Let $U\mathbb{CP}^n = \{(P, \vec{v}) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : \vec{v} \in P\}$, where P is thought of as a complex line in $\mathbb{C}^{n+1}$. In coordinates this is
$$\{\big([p_0 : p_1 : \ldots : p_n], (v_0, v_2, \ldots, v_n)\big) : (v_0, v_2, \ldots, v_n) = \lambda(p_0, p_1, \ldots, p_n), \text{for some } \lambda \in \mathbb{C}\}.$$
Let $\pi : U\mathbb{CP}^n \to \mathbb{CP}^n$ be the projection over the first factor. Let $U_i = \{[p_0 : p_1 : \ldots : p_n] \in \mathbb{CP}^n : p_i \neq 0\}$, $0 \le i \le n$. Find trivializations $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$. Find explicitly the transition functions $\tau_{ij} : U_i \cap U_j \to Gl(1, \mathbb{R}) \equiv \mathbb{R}$. Check that they satisfy the cocycle condition, i.e. $\tau_{ij}\tau_{jk} = \tau_{ik}$. [Note: it is easier to find ϕ_i^{-1} first.]

6. Let $U^{\perp}\mathbb{RP}^n = \{(P, \vec{v}) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : \vec{v} \perp P\}$, where P is thought of as a real line in $\mathbb{R}^{n+1}$. In coordinates this is

$$\{([p_0 : p_1 : \ldots : p_n], (v_0, v_2, \ldots, v_n)) : \sum_{k=0}^{n} p_k v_k = 0\}.$$

Let $\pi : U^{\perp}\mathbb{RP}^n \to \mathbb{RP}^n$ be the projection over the first factor. Let $U_i = \{[p_0 : p_1 : \ldots : p_n] \in \mathbb{RP}^n : p_i \neq 0\}$, $0 \leq i \leq n$. Find trivializations $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$. Find explicitly the transition functions $\sigma_{ij} : U_i \cap U_j \to Gl(n, \mathbb{R})$. Check that they satisfy the cocycle condition, i.e. $\sigma_{ij}\sigma_{jk} = \sigma_{ik}$.

[NOTE: if you want, just do it for $n = 3$ and for the trivializations corresponding to U_0, U_1, U_2. Also, it is easier to find ψ_i^{-1} first.]