Differential Geometry. Homework 4. Due March 3rd. Professor: Luis Fernández

1. Lee, second edition, exercise 15-5 (page 397): Let M be a smooth manifold with or without boundary. Show that the total spaces of TM and T^*M are orientable.

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2. For a hypersurface S in $\mathbb{R}^n$ (and we'll see later also on a manifold), if N is a normal unit vector field along S, the induced volume form determined by N is given by $i_N(dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n)$, where i_N denotes interior multiplication. Show that the induced volume form in S^n when we take N outward pointing is

$$x^1\ dx^2 \wedge dx^3 \wedge \ldots \wedge dx^n - x^2\ dx^1 \wedge dx^3 \wedge \ldots \wedge dx^n + \cdots + (-1)^{n+1} x^n\ dx^1 \wedge dx^2 \wedge \ldots \wedge dx^{n-1}.$$

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3. Lee, second edition, exercise 15-3 (page 397): Suppose $n \geq 1$, and let $\alpha : S^n \to S^n$ be the antipodal map: $\alpha(x) = -x$. Show that α is orientation-preserving if and only if n is odd.
[Hint: The previous exercise gives you an orientation form of S^n.]

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4. Prove that the real projective space $\mathbb{RP}^n$ is orientable if and only if n is odd.
[Hint: Use the previous exercise.]

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5. Recall the classical theorem of Green: If D is a domain in $\mathbb{R}^2$,

$$\oint_{\partial D} (Pdx + Qdy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

Show that it can be deduced from the generalized Stokes theorem: $\displaystyle\int_{\partial D} \omega = \int_D d\omega$.

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6. (Optional - requires some annoying computations). Recall the classical theorem of Stokes: If $\vec{X} \in \mathfrak{X}(\mathbb{R}^3)$, S a surface with smooth boundary in $\mathbb{R}^3$ then

$$\oint_{\partial S} \vec{X} \cdot d\vec{r} = \iint_S \overrightarrow{\text{curl}}(\vec{X}) \cdot \vec{N}\ \ dS$$

where N is an outward-pointing unit normal to S.

Show that it can be deduced from the generalized Stokes theorem: $\displaystyle\int_{\partial D} \omega = \int_D d\omega$.

[Here you need to review some multivariable calculus: recall that $dS = \left| \frac{\partial \phi}{\partial t} \times \frac{\partial \phi}{\partial s} \right| dt\ ds$, where ϕ is a parametrization $\phi(t, s)$ of the surface S. So you need to choose some form ω (not hard if you look at the RHS), take parametrizations on both sides, and check that each side is equal to the corresponding side in the generalized Stokes theorem.]

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7. Recall the classical divergence theorem: If $\vec{X} \in \mathfrak{X}(\mathbb{R}^3)$, V a volume with smooth boundary in $\mathbb{R}^3$ then

$$\iint_{\partial V} \vec{X} \cdot \vec{N}\ \ dS = \iiint_V \text{div}(\vec{X})\ \ dV,$$

where N is an outward-pointing unit normal to V.

Show that it can be deduced from the generalized Stokes theorem: $\displaystyle\int_{\partial D} \omega = \int_D d\omega$.

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8. Warner, Chapter 4, Exercise 12: If α and β are closed differential forms (that is, $d\alpha = d\beta = 0$), prove that $\alpha \wedge \beta$ is closed. If, in addition, β is exact (that is, $\beta = d\gamma$ for some form γ), prove that $\alpha \wedge \beta$ is exact.

9. Lee, second edition, exercise 16-1, page 434: Let $v_1, \ldots, v_n$ be any n linearly independent vectors in $\mathbb{R}^n$, and let P be the n-dimensional parallelepiped they span:

$$P = \{t_1 v_1 + \cdots + t_n v_n : 0 \le t_i \le 1\}.$$

Show that $\mathrm{Vol}(P) = |\det(v_1, \ldots, v_n)|$.

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10. Lee, second edition, exercise 16-2, page 434: Let $T^2 = S^1 \times S^1 \subset \mathbb{R}^4$ denote the 2-torus, defined as the set of points (w, x, y, z) such that $w^2 + x^2 = y^2 + z^2 = 1$, with the product orientation determined by the standard orientation on S^1. Compute $\int_{T^2} \omega$, where ω is the following 2-form on R^4:

$$\omega = xyz \; dw \wedge dy.$$

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11. The natural volume form of $S^{n-1} \subset \mathbb{R}^n$ is given by

$$\alpha_{n-1} = \sum_{i=1}^{n} (-1)^{i+1} x^i \; dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n,$$

Where $(x^1, \ldots, x^n)$ are coordinates in $\mathbb{R}^n$ (see exercise 2). Integrate the form α_{n-1} over S^{n-1} to show that the volume of S^{n-1} is

$$\begin{cases} \dfrac{(2\pi)^{n/2}}{2 \cdot 4 \cdots (n-2)} & \text{if } n \text{ is even} \\ \dfrac{2(2\pi)^{(n-1)/2}}{1 \cdot 3 \cdots (n-2)} & \text{if } n \text{ is odd} \end{cases}$$

[Several hints that can be given, but I give it to you like this so you can think about it. Please do ask if you need.]

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12. (Spherical coordinates in $\mathbb{R}^n$) Consider the map $G : S^{n-1} \times (0, \infty) \to \mathbb{R}^n$ given by $G(p, r) = rp$. Show that $G^*(dx^1 \wedge \cdots \wedge dx^n) = dr \wedge \alpha_{n-1}$, where α_{n-1} is the form of the previous exercise.
Use this fact and the previous exercise to find the volume of the n-ball $B_n = \{p \in \mathbb{R}^n : \|p\| \le 1\}$.

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13. Use Stokes' theorem and exercise 12 to find the volume of the n-ball in a different way.